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EQUIVALENT BANDWIDTH OF A GENERAL CLASS OF POLYNOMIAL SMOOTHERS--ETC(U)
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NUSC Technical Report 6601

19 July 1982

Equivalent Bandwidth of a General Class of Polynomial Smoothers: With Application to Bearing Tracker Random Error Evaluation

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Preface

This report was prepared under NUSC Project No. A18002. The Principal Investigator is Kenneth Sargent (Code 3291). The sponsoring activity is NAVSEA, Program Manager Dr. Robert Snuggs (NAVSEA PMS 409C).

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REVIEWED AND APPROVED: 19 July 1982

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TR 6601	2. GOVT ACCESSION NO. AD-A117 996	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) EQUIVALENT BANDWIDTH OF A GENERAL CLASS OF POLYNOMIAL SMOOTHERS: WITH APPLICATION TO BEARING TRACKER RANDOM ERROR EVALUATION		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Robert A. LaTourette Lawrence C. Ng		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Underwater Systems Center New London Laboratory New London, Connecticut 06320		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS A18002
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE 19 July 1982 13. NUMBER OF PAGES 82
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Random bearing error is a major performance measure of a sonar bearing tracker. Programs currently employed in calculating random bearing error from measured tracker bearing error data use a standard polynomial Least Mean Square Fit (LMSF) algorithm to remove an unknown time varying mean. Previously, the effect of the LMSF algorithm on the residuals of the measured tracker bearing error data was not fully accounted for. In addition, when processing correlated bearing error residuals, the optimum choice of the order of the LMSF and appropriate bias correction factor as a function . (over)

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of signal-to-noise ratio (SNR) were not known.

This study investigates the properties of the LMSF in detail and shows that the LMSF behaves as a low-pass filter, the frequency response characteristics of which can be calculated exactly. The equivalent noise bandwidth of the LMSF is shown to be a function of the sample size, the sampling time and the order of the fit. The appropriate bias correction factor, when processing correlated data, is shown to be determined by the ratio of the LMSF bandwidth to the equivalent tracker bandwidth. Results of the analysis are verified by extensive simulation. Finally, an operational procedure is given to obtain an unbiased estimate of the variance for at-sea measured tracker data.

TABLE OF CONTENTS

	Page
LIST OF FIGURES	iii/iv
1.0 INTRODUCTION	1
1.1 DESCRIPTION OF PREVIOUS RANDOM ERROR CALCULATION TECHNIQUE	1
1.2 PROBLEMS WITH THE PREVIOUS RANDOM ERROR CALCULATION TECHNIQUE	2
1.3 TECHNICAL OBJECTIVE	3/4
2.0 THEORETICAL ANALYSIS OF LMSF PROCEDURE	5
2.1 THEORETICAL MODELS	5
2.2 EQUIVALENT LMSF BANDWIDTH CORRECTION -- FREQUENCY DOMAIN APPROACH	6
2.3 EQUIVALENT LMSF BANDWIDTH CORRECTION -- ANALYTICAL TIME DOMAIN APPROACH	11
2.3.1 Unbiased Variance Estimate from Stationary Correlated Data	12
2.3.2 Unbiased Variance Estimate from Non-Stationary White Data	16
2.3.3 Equivalent Bandwidth of LMSF in the Presence of White Data	22
2.3.4 Unbiased Variance Estimate from Non-Stationary Correlated Data	27
2.3.5 Equivalent Transfer Function of LMSF in the Presence of Correlated Data	34
2.3.6 Alternate Expression of the Exact Correction Factor	38
2.4 ERROR ANALYSIS OF THE UNKNOWN MEAN MODEL	40
3.0 COMPUTER-GENERATED DATA	45
3.1 THEORETICAL DATA	45
3.2 SIMULATION DATA	48
4.0 NEW RANDOM ERROR CALCULATION TECHNIQUE	51
5.0 SUMMARY AND CONCLUSIONS	55/56

TABLE OF CONTENTS (Cont'd)

		Page
APPENDIX A	PROOF OF $\mathrm{sum}[H(H^TH)^{-1}H^T] = N$	A-1
APPENDIX B	PROOF OF $\mathrm{sum}[H(H^TH)^{-1}H^TRH(H^TH)^{-1}H^T] = \mathrm{sum}[R]$	B-1
APPENDIX C	EQUIVALENT BANDWIDTH OF A FIRST ORDER DIGITAL FILTER	C-1
REFERENCES	. . . .	R-1

LIST OF FIGURES

Figure		Page
1	Illustration of Input Variance to Output Variance Relationship	57
2	LMSF Power vs Frequency. Polynomial Fit 100 Samples. . .	58
3	LMSF Power vs Frequency. 4th Order sin/cos Fit 100 Samples.	59
4	LMSF Power vs Frequency. 6th Order sin/cos Fit 100 Samples.	60
5	LMSF Power vs Frequency. 6th Order Polynomial Fit. . . .	61
6	% Error of STD and New Correction Factor vs Ratio of Bandwidths. 0th Order Polynomial Fit 100 Samples. . . .	62
7	% Error of STD and New Correction Factor vs Ratio of Bandwidths. 3rd Order Polynomial Fit 100 Samples. . . .	63
8	% Error of STD and New Correction Factor vs Ratio of Bandwidths. 6th Order Polynomial Fit 100 Samples. . . .	64
9	LMSF Power vs Frequency. 3rd Order Polynomial Fit 100 Samples.	65
10	LMSF Power vs Frequency. 6th Order Polynomial Fit 100 Samples.	66
11	LMSF Power vs Frequency. 3rd Order Polynomial Fit 64 Samples.	67
12	LMSF Power vs Frequency. 6th Order Polynomial Fit 64 Samples.	68
13	% Error of New Correction Factor vs Ratio of Bandwidths. 6th Order Polynomial Fit 128 Samples.	69
14	Comparison of New vs Old Random Error Techniques	70

1.0 INTRODUCTION

Random bearing error is defined as the standard deviation of bearing error fluctuation about the mean bearing error. Random bearing error is obtained from recorded at-sea test data by analyzing bearing error data recorded over time at a 1-second rate while keeping relevant parameters, such as signal-to-noise ratio (SNR) and actual relative bearing, stationary. In practice, the above parameters are not constant. In particular, the actual relative bearing and thus the mean bearing error vary as a function of time. The time-varying mean must be removed prior to the variance calculation. Since no statistical knowledge of the time varying mean is assumed, a standard trend removal procedure of using a polynomial Least Mean Square Fit (LMSF) is used. The LMSF procedure causes bias in the resulting variance estimate; thus, in order to obtain an unbiased estimate, the classical LMSF correction factor is applied to the sum of the squared residuals. However, this correction is not adequate when the measured tracker bearing error data is significantly correlated. This is the case when tracking in a low SNR environment.

This report presents a detailed study on the LMSF procedure, and provides an appropriate bias correction for the variance estimate based on the input data bandwidth when processing correlated data.

1.1 DESCRIPTION OF PREVIOUS RANDOM ERROR CALCULATION TECHNIQUE

The previous method for calculating random bearing error assumed the following input bearing error model:

$$\text{Bearing Error} = m(t) + n(t) \tag{1}$$

where n(t) is the desired tracker random bearing error output with the following assumed properties;

1. White sequence (over the sampling time of 1 second)
2. Zero mean

and m(t) is an unknown time varying mean with a time constant much longer than 1 second.

The procedure for calculating random bearing error consisted of the following:

1. Select time intervals between 60 and 400 seconds, where the indicated SNR remains relatively constant and tracker bearing error outputs indicate normal automatic tracking.

2. Perform a 6th order polynomial LMSF on the selected data. (For low SNR use a 0th order polynomial fit.)

3. Calculate random bearing error from the sum of the squared residuals (Σ^2) of the LMSF by using the following formula:

$$\sigma^2_{\text{Bearing}} = \frac{\Sigma^2}{N - K - 1} \qquad (2)$$

where N = the number of samples
K = the order of the fit
Σ^2 = the sum of the squared residuals.

1.2 PROBLEMS WITH THE PREVIOUS RANDOM ERROR CALCULATION TECHNIQUE

The major assumption of the previous random error calculation technique is that the polynomial LMSF removes the unwanted time varying mean m(t) while retaining the desired unbiased random tracker bearing error output n(t). The key assumption is that n(t) is effectively a white sequence for the given

sampling time of 1 second. In reality, the above assumption is seldom true. However, at relatively high SNRs where the tracker averaging time is small, the incurred bias is relatively small. At very low SNRs, where the tracker averaging time becomes much longer, the biases will become substantial. The technique of reducing the order of the fit to zero was necessary to reduce this bias to an acceptable level.

1.3 TECHNICAL OBJECTIVE

The LMSF procedure can be viewed as a low-pass filter. If the equivalent cut-off frequency of the LMSF procedure could be determined as a function of its parameters, then intelligent decisions could be made on selecting the input parameters and determining the appropriate bias correction factors. In the following sections, a model relating the LMSF procedure to a low-pass filter will be developed and analyzed.

2.0 THEORETICAL ANALYSIS OF LMSF PROCEDURE

In this section, a description of the LMSF procedure as a low-pass filter shall be developed under the assumption of a zero-mean, white input sequence. The derived model of the LMSF as a low-pass filter shall then be applied to the case of a zero-mean correlated input sequence. For the correlated case, an excellent approximation shall be derived by correcting the resulting "sum of squared residuals" of the LMSF fit to an unbiased estimator of the act l input variance, σ_A^2. The above approximation assumes that the equivalent nd-width of the LMSF is much smaller than the equivalent bandwidth of the c - lated input sequence. This relationship is shown as

$$BW_{LMSF} << BW_{Input} \leq \frac{1}{\Delta t} \qquad (3)$$

where Δt = sampling time in seconds.

2.1 THEORETICAL MODELS

The assumed model for the bearing error shall initially be the same as Equation (1).

Bearing Error = m(t) + n(t)

where n(t) is the desired tracker random bearing error output with the following assumptions:

1. White sequence (over the sampling time of Δt seconds)
2. Zero mean
3. Variance = σ_A^2

and m(t) is a deterministic but unknown time-varying mean with a very long time constant compared to the sampling time.

The LMSF can be modeled as a linear process having the following properties:

Let RLMSF be defined as the "sum of squared residuals" of the LMSF.

Then

$$\mathrm{RLMSF}(m(t) + n(t)) = \mathrm{RLMSF}(m(t)) + \mathrm{RLMSF}(n(t)) \tag{4}$$

provided that the zero mean noise process $n(t)$ is uncorrelated with the time varying mean $m(t)$.

The only restrictions concerning the form of the fitting function are that it be selected from a set of complete functionals and that a constant term be included.

Since the LMSF is a linear process, its effect on $m(t) + n(t)$ can be analyzed separately. Considering the effect of the LMSF on the unwanted time-varying mean separately would dictate that a higher order fit be used in order to remove the unwanted mean completely. Unfortunately, this also removes a portion of the noise process. With this in mind, we shall now look at the effect of the LMSF on the input noise process, $n(t)$. In a later section we shall return to analyze the choice and effect of the LMSF procedure on the unwanted mean, $m(t)$.

2.2 EQUIVALENT LMSF BANDWIDTH CORRECTION - FREQUENCY DOMAIN APPROACH

Let $|H(f)|^2$ be the power transfer function of a digital low-pass filter, operating at a sampling interval of Δt seconds. Let σ_A^2 be the variance (or power) of a zero-mean white sequence. The expected power spectral density ($\overline{SPD}$) of the above process, sampled at intervals of Δt seconds is given by:

$$\overline{SPD} = \sigma_A^2 \Delta t \tag{5}$$

which is uniform over the frequency band.

The expected power $\overline{\sigma^2_{fil}}$ at the output of the low-pass filter, given the above zero-mean white sequence, is given by:

$$\overline{\sigma^2_{fil}} = \int_0^{\frac{1}{\Delta t}} \frac{|H(f)|^2}{|H(0)|^2} \cdot \frac{\sigma_A^2}{\frac{1}{\Delta t}} \, df$$

or

$$\overline{\sigma^2_{fil}} = \sigma_A^2 \cdot \Delta t \cdot \frac{1}{|H(0)|^2} \cdot \int_0^{\frac{1}{\Delta t}} |H(f)|^2 df \qquad (6)$$

However, the equivalent two-sided bandwidth (BW_e) can be written as

$$BW_e = \frac{1}{|H(0)|^2} \int_0^{\frac{1}{\Delta t}} |H(f)|^2 df \qquad (7)$$

Note that we have implicitly assumed in Equation (7) that the input sequence is sampled at or above the Nyquist rate.

Therefore, combining Equation (6) and Equation (7) yields

$$\overline{\sigma^2_{fil}} \approx \sigma_A^2 \cdot \Delta t \cdot BW_e \qquad (8)$$

Let $\overline{\Sigma^2}$ be the expected value of the "sum of squared residuals" of a kth order LMSF to a white sequence of variance, σ_A^2. The input sequence is sampled N times at intervals of Δt seconds. The LMSF function contains K + 1 terms. Also, let the expected standard variance $\overline{\sigma_S^2}$ be defined as:

$$\overline{\sigma_S^2} = \frac{\overline{\Sigma^2}}{N} \qquad (9)$$

However, regression theory [1] states that

$$\frac{\Sigma^2}{N - K - 1} \text{ is an unbiased estimator of } \sigma_A^2;$$

therefore

$$\sigma_A^2 = \frac{\overline{\Sigma^2}}{N - K - 1} \tag{10}$$

The expected variance $\overline{\sigma^2_{fil}}$ of the LMSF filter with respect to the white input sequence can be written as

$$\overline{\sigma^2_{fil}} = \sigma_A^2 - \overline{\sigma_S^2} \tag{11}$$

Figure 1 illustrates the above equation.

Using Equation (9) and Equation (10) in Equation (11) yields

$$\overline{\sigma^2_{fil}} = \sigma_A^2 \left[1 - \frac{N - K - 1}{N}\right]$$

or

$$\overline{\sigma^2_{fil}} = \sigma_A^2 \cdot \frac{K + 1}{N} \tag{12}$$

Equating Equation (8) with Equation (12) yields an equation for the (two-sided) equivalent bandwidth for the LMSF low-pass filter, specifically;

$$BW_e(\text{LMSF}) = \frac{K + 1}{N \cdot \Delta t} \tag{13}$$

Note that the only assumptions made relating to the form of the fitting functions were that they were selected from a functionally complete set and that they include a constant term. This implies that the effect on the variance estimate from the input white sequence, within the above restriction, is on the average exactly predictable and independent of the choice of fitting functions. However, the choice of fitting functions can have a drastic effect on the removal of the unwanted mean m(t). The above property can be used to advantage, as shall be discussed later.

Let us now assume we have a correlated, zero-mean input sequence. We shall model the above sequence as the output of a discrete low-pass filter, with sampling time Δt seconds and driven by a unit variance zero-mean white input sequence. If the power spectral density function of the low-pass filter is given by $|W(f)|^2$, then the variance of the correlated sequence, σ_A^2, is

$$\sigma_A^2 = \int_0^{\frac{1}{\Delta t}} |W(f)|^2 df \tag{14}$$

The (two-sided) equivalent bandwidth of the input correlated sequence is defined by

$$BW_e(\text{Input}) = \frac{1}{|W(0)|^2} \int_0^{\frac{1}{\Delta t}} |W(f)|^2 df \tag{15}$$

It is assumed that

$$BW_e(\text{LMSF}) << BW_e(\text{Input}) \leq \frac{1}{\Delta t} \tag{16}$$

The objective is to find a correction factor, C, relating the expected value of the standard variance ($\overline{\sigma_S^2} = \overline{\Sigma^2}/N$) of the LMSF to the actual variance σ_A^2 of the correlated zero-mean input sequence

$$\sigma_A^2 = C \cdot \overline{\sigma_S^2} \tag{17}$$

The expected measured variance, $\overline{\sigma_S^2}$, at the output of a low-pass filter H(f) for the given correlated zero-mean input sequence can be written as

$$\overline{\sigma_S^2} = \sigma_A^2 - \int_0^{\frac{1}{\Delta t}} \frac{|H(f)|^2}{|H(0)|^2} \cdot |W(f)|^2 df \tag{18}$$

or, using Equation (14):

$$\overline{\sigma_S^2} = \sigma_A^2 \cdot \left[1 - \frac{\int_0^{\frac{1}{\Delta t}} \frac{|H(f)|^2}{|H(0)|^2} \cdot |W(f)|^2 df}{\int_0^{\frac{1}{\Delta t}} |W(f)|^2 df} \right] \tag{19}$$

Substituting Equation (15) into Equation (19) yields

$$\overline{\sigma_S^2} = \sigma_A^2 \cdot \left[1 - \frac{\frac{1}{|H(0)|^2} \int_0^{\frac{1}{\Delta t}} \frac{|W(f)|^2}{|W(0)|^2} \cdot |H(f)|^2 df}{BW_e(\text{Input})} \right] \tag{20}$$

Assuming that: 1) $|W(f)|^2$ is reasonably flat at low frequencies, 2) roll-off of $|H(f)|^2$ is rapid, and 3) the equivalent bandwidth of $|H(f)|^2$ is much smaller than the equivalent bandwidth of $|W(f)|^2$, the above integral can be approximated by the following:

$$\frac{1}{|H(0)|^2} \int_0^{\frac{1}{\Delta t}} \frac{|W(f)|^2}{|W(0)|^2} \cdot |H(f)|^2 df \approx \frac{1}{|H(0)|^2} \int_0^{\frac{1}{\Delta t}} |H(f)|^2 df = BW_e(\text{fil}) \tag{21}$$

Note that if $|H(f)|^2 = \delta(f)$, an impulse response in frequency (which is the case for a zero-order LMSF), then the above approximation is exact.

Substituting this result into Equation (20) yields

$$\overline{\sigma_S^2} \approx \sigma_A^2 \cdot \left[1 - \frac{BW_e(\text{fil})}{BW_e(\text{Input})} \right]$$

or

$$\sigma_A^2 \approx \left[\frac{1}{1 - \frac{BW_e(fil)}{BW_e(Input)}} \right] \cdot \overline{\sigma_S^2} \tag{22}$$

Substituting the equivalent bandwidth of the LMSF filter $BW_e(LMSF)$, Equation (13) for the equivalent bandwidth of the low pass filter, $BW_e(fil)$, yields

$$\sigma_A^2 \approx \left[\frac{1}{1 - \frac{K + 1}{N \cdot \Delta t \cdot BW_e(Input)}} \right] \cdot \overline{\sigma_S^2} \tag{23}$$

Finally, comparing Equation (23) and Equation (17) we have the desired approximation to the correction factor, C.

$$C \approx \frac{1}{1 - \frac{K + 1}{N \cdot \Delta t \cdot BW_e(Input)}} \tag{24}$$

Computer analysis has indicated that the polynomial LMSF filter does have a reasonable roll-off of approximately 6 dB/octave. Therefore, if the low-pass filter model for the input correlated noise sequence is reasonable, then the above approximation is very good for $BW_e(LMSF) << BW_e(Input) \leq 1/\Delta t$. As noted earlier, for the case where $K = 0$ the approximation is exact.

2.3 EQUIVALENT LMSF BANDWIDTH CORRECTION -- ANALYTICAL TIME DOMAIN APPROACH

Guided by the intuitive approach as described in Section 2.2, this section analytically re-examines the problem of obtaining an unbiased estimate of the variance from finite observations (or measurements) of a non-stationary, correlated, random process. For the most part, the analyses shown in this section support the intuitive results as presented in Section 2.2. The developments in this section show that: 1) LMSF behaves as a low-pass filter whose bandwidth is determined by the triplet (K, N, Δt) where K is the order of the LMSF, N is the number of samples, and Δt is the sampling time; and 2) correlated noise input and the subsequent application of the LMSF to remove

non-stationary mean results in biases in the variance estimates. The analysis shows that in order to obtain an unbiased estimate of the variance, a correction factor which accounts for the correlated data and the effect of the LMSF is needed.

The following analyses assume no statistical knowledge of the non-stationary mean, except that it is a slowly varying function of time and is representable by a finite order set of functions (for example, polynomials). In order to provide a proper perspective of the analysis, the unbiased estimate of the variance of a stationary process with correlated data sequence will be discussed first. After an unbiased correction factor has been derived, we then consider the further complication of non-stationary mean.

2.3.1 Unbiased Variance Estimate from Stationary Correlated Data

Let $x_1, x_2, \ldots, x_N$ be a sequence of N observations (or measurements) from a random process, the mean and covariance of which are given by

$$\begin{aligned} &E[x_i] = m \qquad i = 1, 2, \ldots, N \\ &E[(x_i - m)(x_j - m)] = \sigma^2\rho_{ij} \quad \forall\ i, j \end{aligned} \tag{25}$$

where ρ_{ij} is the normalized covariance coefficient between samples x_i and x_j. What is now sought is an unbiased estimate of the mean m and the variance σ^2 of the random process from the observed sequence $\{x_n\}$, n = 1, 2, ..., N. If the data is uncorrelated; i.e., $\rho_{ij} = \delta_{ij}$, then it is well known [2] that the standard unbiased estimates of the mean and variance are given by:

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{26}$$

$$\hat{\sigma}^2 = \mu_0 \hat{\sigma}_S^2 \tag{27}$$

where

$$\hat{\sigma}_S^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \overline{x})^2 \tag{28}$$

and

$$\mu_0 = \frac{1}{1 - \frac{1}{N}} \tag{29}$$

are defined as the standard estimate of the variance and the required unbiased estimate correction factor, respectively. It will be shown later that μ_0 is the unbiased estimate correction factor for a zero-order LMSF.

However, if the data is correlated, as is the case at present, the estimated mean $\overline{x}$ remains unbiased whereas the variance becomes biased. Furthermore, Equation (27) always underestimates the true variance. This can be shown readily as follows:

Taking the expectation of Equation (27), we obtain

$$E[\hat{\sigma}^2] = \mu_0 E[\hat{\sigma}_S^2]$$

$$= \mu_0 E\left\{\frac{1}{N} \sum_{i=1}^{N} [(x_i - m) - (\overline{x} - m)]^2\right\}$$

$$= \mu_0 E\left\{\frac{1}{N} \sum_{i=1}^{N} (x_i - m)^2 - \frac{2}{N} \sum_{i=1}^{N} (x_i - m)(\overline{x} - m) + \frac{1}{N} \sum_{i=1}^{N} (\overline{x} - m)^2\right\}$$

$$= \mu_0 \left\{ \sigma^2 - \frac{\sigma^2}{N^2} \sum_i \sum_j \rho_{ij} \right\}$$

$$= \mu_0 \left(1 - \frac{\sum_i \sum_j \rho_{ij}}{N^2} \right) \sigma^2 \qquad \text{for all i and j.} \tag{30}$$

The effective number of independent samples are defined as

$$N_e \triangleq \frac{N^2}{\sum_i \sum_j \rho_{ij}} \tag{31}$$

It can now be seen that

$$N_e = \begin{cases} N \text{ if } \rho_{ij} = \delta(i - j) \ \forall \ i, j,; \text{ i.e., uncorrelated} \\ 1 \text{ if } \rho_{ij} = 1 \ \forall \ i, j,; \text{ i.e., highly correlated} \end{cases} \tag{32}$$

Therefore, Equation (30) can be written as:

$$E[\hat{\sigma^2}] = \mu_0 \left(1 - \frac{1}{N_e} \right) \sigma^2$$

$$= \left(\frac{1 - \frac{1}{N_e}}{1 - \frac{1}{N}} \right) \sigma^2 \tag{33}$$

Since from Equation (32) $N_e \lesssim N$, we have

$$\left(\frac{1 - \frac{1}{N_e}}{1 - \frac{1}{N}} \right) \leq 1$$

thus, in general

$$E[\hat{\sigma}^2] \leq \sigma^2$$

and

$$E[\hat{\sigma}^2] = \sigma^2 \tag{34}$$

if, and only if, the data sequence is uncorrelated (or white).

Thus, it is shown that the standard unbiased estimate of variance from a correlated data sequence becomes biased and the true variance is always underestimated. The unbiased estimate of the variance is defined as follows:

$$\hat{\sigma}^2_{ub} = \left(\frac{1 - \frac{1}{N}}{1 - \frac{1}{N_e}} \right) \hat{\sigma}^2$$

$$= \nu_0 \hat{\sigma}^2_S \tag{35}$$

where

$$\nu_0 = \frac{1}{1 - \frac{1}{N_e}} \tag{36}$$

is the unbiased correction factor.

If the equivalent bandwidth of the correlated noise is defined as B_N, then for T second observation time, we have

$$N_e = TB_N$$

$$= N \cdot \Delta t \cdot B_N \tag{37}$$

and so the unbiased correction factor becomes:

$$\nu_0 = \frac{1}{1 - \frac{1}{N \cdot \Delta t \cdot B_N}}$$

$$= \frac{1}{1 - \frac{B_f}{B_N}} \qquad (38)$$

where $B_f \triangleq \frac{1}{N\Delta t}$ is the equivalent bandwidth of the zero-order LMSF, as will be shown later.

This is identical to Equation (24) where a zero-order fit is assumed. Thus Equation (38) indicates that if one can estimate the equivalent bandwidth of the random process and calculate the equivalent bandwidth of the LMSF, then an approximate unbiased correction factor on the standard variance calculation can be found. Note that in the following sections the unbiased correction factors are denoted by either μ_k or ν_k, where μ_k is the correction factor for the kth order fit with white noise, and ν_k is the correction factor for the kth order fit with correlated noise.

2.3.2 Unbiased Variance Estimate from Non-Stationary White Data

It is assumed the non-stationary component of the random process is due to the time varying mean. It is further assumed that the mean variation can be modeled by a finite set of functions in time. The LMSF technique is used to remove the non-stationary component. The noise process is assumed white, and an unbiased estimate of the variance from a finite sequence of observations is sought.

It will be shown that LMSF behaves as a low-pass filter, the filter frequency response of which can be controlled by the triplet (K, N, Δt). An expression will be derived which indicates the resulting accuracy of the variance estimate.

Let $\{x_i\}$, i = 1, 2, ..., N be a sequence of N observations from a random process, the mean and variance of which are given by:

$$E[x_i] = m_i$$

$$E[(x_i - m_i)(x_j - m_j)] = \sigma^2\delta_{ij} \tag{39}$$

where δ_{ij} is the kronecka delta function. It is assumed that m_i can be modeled by a finite order polynomial; namely:

$$m(t) = \sum_{k=0}^{K} a_k t^k$$

or for discrete time

$$m_i = \sum_{k=0}^{K} a_k t_i^k \qquad i = 1, 2, \ldots, N \tag{40}$$

Note that for convenience we have modeled the time varying mean by polynomials. However, the results presented herein are applicable to any other polynomials. For example, one obtains a trigonometric polynomial based on a Fourier series representation.

Now let

$$\underline{t}_i = [1 \; t_i \; t_i^2 \; \ldots \; t_i^k]^T,$$

$$\underline{a} = [a_0 \; a_1 \; a_2 \; \ldots \; a_N]^T \tag{41}$$

where $[\;]^T$ denotes the transpose of a vector.

Then, the time varying mean, Equation (40) can be written in vector form as:

$$m_i = \underline{t}_i^T\underline{a} \tag{42}$$

The N observations can be written as:

$$x_i = m_i + w_i;\ i = 1, 2, \ldots, N \tag{43}$$

where

$$w_i \triangleq x_i - m_i$$

with

$$E[w_i] = 0,\ E[w_i w_j] = \sigma^2\delta(i - j)$$

Equation (43) can be written in matrix form as:

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} = \begin{bmatrix} 1 & t_1 & \ldots & t_1^k \\ 1 & t_2 & \ldots & t_2^k \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 1 & t_N & \ldots & t_N^k \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_k \end{bmatrix} + \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ w_N \end{bmatrix} \tag{44}$$

or more concisely as:

$$\underline{X} = \underline{M} + \underline{W} \tag{45}$$

with

$$\underline{M} = H\underline{a}$$

where H is a matrix of dimension Nx(K + 1).

The parameters of interest are the unknown coefficients, $\underline{a}$. The method of least square [4] minimizes the following quadratic expression:

$$J = (\underline{X} - H\underline{a})^T(\underline{X} - H\underline{a})$$

with respect to the unknown coefficient vector, $\underline{a}$. The resulting solution is given by:

$$\hat{\underline{a}} = L\underline{X}; \text{ where } L = (H^TH)^{-1}H^T \tag{46}$$

and L is a known linear operator.

Using Equation (45), Equation (46) can be written as:

$$\hat{\underline{a}} = L\underline{M} + L\underline{W} \tag{47}$$

Thus, the LMSF is a linear process.

Since $\hat{\underline{a}}$ depends on the noise vector $\underline{W}$, $\hat{\underline{a}}$ is a random vector. However, $\hat{\underline{a}}$ is an unbiased estimate of $\underline{a}$ since (from Equation (47)):

$$\begin{aligned} E[\hat{\underline{a}}] &= L\underline{M} \\ &= (H^TH)^{-1}H^TH\underline{a} \\ &= \underline{a} \end{aligned} \tag{48}$$

Furthermore, the covariance of $\hat{\underline{a}}$ is given by:

$$E[(\hat{\underline{a}} - \underline{a})(\hat{\underline{a}} - \underline{a})^T] = E[L\underline{W}\underline{W}^TL^T] = (H^TH)^{-1}\sigma^2 \tag{49}$$

The estimated mean is given by:

$$\hat{\underline{M}} = H\hat{\underline{a}}$$

and the error residual is:

$$\underline{e} = \underline{X} - \underline{\hat{M}}$$

$$= H(\underline{a} - \underline{\hat{a}}) + \underline{W} \tag{50}$$

The standard estimate of the variance is given by the averaged sum of the square of the residuals; i.e.:

$$\hat{\sigma}_S^2 = \frac{1}{N} \sum_{i=1}^{N} e_i^2$$

$$= \frac{1}{N} \text{Trace}(\underline{ee}^T)$$

$$= \frac{1}{N} \text{Trace}\{H(\underline{a} - \underline{\hat{a}})(\underline{a} - \underline{\hat{a}})^T H^T + \underline{WW}^T + H(\underline{a} - \underline{\hat{a}})\underline{W}^T + \underline{W}(\underline{a} - \underline{\hat{a}})^T H^T\} \tag{51}$$

where for a given matrix, A, Trace(A) $\stackrel{\Delta}{=} \sum_{i=1}^{N} a_{ii}$.

In order to evaluate the quality (amount of bias) of the estimate given by Equation (51), the expectation on both sides of the equation is taken, which yields:

$$E[\hat{\sigma}_S^2] = \frac{1}{N} \text{Trace}\{HE[(\underline{a} - \underline{\hat{a}})(\underline{a} - \underline{\hat{a}})^T]H^T + E[\underline{WW}^T] + HE[(\underline{a} - \underline{\hat{a}})\underline{W}^T] + \ldots$$

$$+ E[\underline{W}(\underline{a} - \underline{\hat{a}})^T H^T]\} \tag{52}$$

Recall from Equation (47) that:

$$\underline{\hat{a}} = (H^T H)^{-1} H^T [\underline{M} + \underline{W}]$$

$$= \underline{a} + (H^T H)^{-1} H^T \underline{W}$$

then the following relations can be derived:

$$
\begin{aligned}
E[(\underline{a} - \hat{\underline{a}})(\underline{a} - \hat{\underline{a}})^T] &= (H^T H)^{-1}\sigma^2 \\
E[\underline{W}\underline{W}^T] &= \sigma^2 I_N \\
E[(\underline{a} - \hat{\underline{a}})\underline{W}^T] &= -(H^T H)^{-1}H^T\sigma^2 \\
E[\underline{W}(\underline{a} - \hat{\underline{a}})^T] &= -H(H^T H)^{-1}\sigma^2
\end{aligned} \tag{53}
$$

Using the relations of Equation (53) in Equation (52) yields:

$$E[\hat{\sigma}_S^2] = \frac{\sigma^2}{N} \operatorname{Trace}\{I_N - H(H^T H)^{-1}H^T\} \tag{54}$$

but

$$
\begin{aligned}
\operatorname{Trace}\{I_N\} &= N \\
\operatorname{Trace}\{H(H^T H)^{-1}H^T\} &= \operatorname{Trace}\{H^T H(H^T H)^{-1}\} \\
&= \operatorname{Trace}\{I_{K+1}\} = K + 1
\end{aligned}
$$

where I_{K+1} denotes the identity matrix of dimension K + 1, and K is the order of the LMSF. Finally, because the Trace operator is a linear operator Equation (54) becomes:

$$E[\hat{\sigma}_S^2] = \frac{N - K - 1}{N} \sigma^2$$

We form the unbiased Kth order fit estimates by forming:

$$
\begin{aligned}
\hat{\sigma}_{ubK}^2 &= \frac{N}{N - K - 1} \hat{\sigma}_S^2 \\
&= \mu_K \sigma_S^2
\end{aligned} \tag{55}
$$

where

$$\mu_K = \frac{1}{1 - \frac{K + 1}{N}} \tag{56}$$

is the unbiased correction factor for the Kth order fit with uncorrelated data. It will be shown in the next section that $(K + 1)/(N\Delta t)$ is simply the equivalent bandwidth of the Kth order LMSF. Note that Equation (56) is the well-known result found in the study of the analysis of the variance [1]. With $K = 0$ (zero-order LMSF) Equation (56) reduces to Equation (29).

2.3.3 Equivalent Bandwidth LMSF in the Presence of White Data

It is well understood that an LMSF behaves as a low-pass filter. However, not so obvious are its frequency response characteristics. This section derives the frequency response of the LMSF. In particular, the relationship of the LMSF with the triplet $(K, N, \Delta t)$ will be analyzed. The equivalent bandwidth (two-sided) of the LMSF is derived for the case of a white noise sequence.

Our approach to calculating the LMSF equivalent bandwidth is as follows:

1. Treat the LMSF as a classical filter, the transfer function of which is sought to be identified.

2. Observe the spectral response of the output sequence when the LMSF filter is driven by a white input sequence.

3. Take the ratio of the output spectral response to the input spectral response yielding the magnitude square response of the LMSF filter.

4. Integrate the magnitude squared response to yield the equivalent bandwidth.

It was shown in the last section that LMSF yields the estimated mean as:

$$\underline{\hat{M}} = H\underline{\hat{a}}$$

where the vectors $\underline{\hat{M}} = [\hat{m}_1, \hat{m}_2, \ldots, \hat{m}_N]^T$ and $\underline{\hat{a}} = (H^TH)^{-1}H^T\underline{X}$

Now let $\hat{M}(\ell)$ denote the Discrete Fourier Transform (DFT) of the time sequence $\{\hat{m}_n\}$; n = 1, 2, ..., N, then:

$$\hat{M}(\ell) = \sum_{n=1}^{N} \hat{m}_n e^{-j\frac{2\pi\ell n}{N}}, \quad \ell = 0, 1, \ldots, N-1 \tag{57}$$

For a zero-mean white sequence, we have:

$$\underline{X} = \underline{W}$$

and the elements of the estimated mean vector, $\underline{\hat{M}}$, become

$$\hat{m}_n = \underline{t}_n^T\underline{\hat{a}} = \underline{t}_n^T L\underline{W}$$

where, it can be recalled, that $\underline{t}_n^T = [1 \; t_n \; \ldots \; t_n^K]$ and $L = (H^TH)^{-1}H^T$, a square symmetric matrix of dimension (K + 1)

Define $P(\ell)$ as the average magnitude square response of the LMSF filter output at frequency ℓ, then:

$$P(\ell) = E\{|\hat{M}(\ell)|^2\} \; ; \; \ell = 0, 1, \ldots, N-1. \tag{58}$$

and $P(\ell)$ is simply the power spectral response of the LMSF filter output.

Using Equation (57), $P(\ell)$ can be expressed as:

$$P(\ell) = E\{|\hat{M}(\ell)|^2\}$$

$$= E\left\{\sum_{n}^{N} \sum_{m}^{N} (\underline{t}_n^T L \underline{W}\underline{W}^T L \underline{t}_m) e^{-j \frac{2\pi\ell}{N}(n - m)}\right\}$$

$$= \sigma^2 \sum_{n}^{N} \sum_{m}^{N} \alpha_{nm} e^{-j \frac{2\pi\ell}{N}(n - m)} \; ; \; \ell = 0, 1, 2, \ldots, N - 1 \tag{59}$$

where

$$\alpha_{nm} = \underline{t}_n^T (H^T H)^{-1} \underline{t}_m .$$

If desired, α_{nm} can be factored as described in the following paragraphs.

Since $(H^T H)^{-1}$ is a square non-singular matrix, it can be factored into a product of two matrices; i.e., one can write

$$(H^T H)^{-1} = D^T D$$

where D is an upper triangular matrix. Therefore:

$$\alpha_{nm} = \underline{t}_n^T D^T D \underline{t}_m$$

$$= \beta_n^T \beta_m \tag{60}$$

where the column vector $\beta_m \stackrel{\Delta}{=} D\underline{t}_m$.

Since the average input spectrum response is $E\{|X(\ell)|^2\} = N\sigma^2$ for all frequency ℓ, the average magnitude square response of the LMSF filter is:

$$|G(\ell)|^2 \triangleq \frac{P(\ell)}{N\sigma^2} = \frac{1}{N}\sum_{n}^{N} \sum_{m}^{N} \beta_n\beta_m e^{-j\,\frac{2\pi\ell}{N}(n-m)}$$

$$= \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \beta_n e^{-j\,\frac{2\pi\ell n}{N}} \right|^2$$

or

$$G(\ell) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \beta_n e^{-j\,\frac{2\pi\ell n}{N}} \;;\; \ell = 0, 1, 2, \ldots, N-1 \tag{61}$$

where $G(\ell)$ is the equivalent transfer function of the LMSF filter.

The equivalent bandwidth of the LMSF filter can be determined by integrating the filter power spectrum and dividing by the filter maximum power level. Thus,

$$B_f \triangleq \frac{\sum_{\ell=0}^{N-1} |G(\ell)|^2}{|G(0)|^2} \tag{62}$$

$G(0)$ is the maximum since

$$|G(\ell)| = \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \beta_n e^{-j\,\frac{2\pi\ell n}{N}} \right| \leq \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |\beta_n| = |G(0)| \quad \forall\ell.$$

Now applying Parseval's Relation [3] yields:

$$\sum_{\ell=0}^{N-1} |G(\ell)|^2 = \frac{1}{N\sigma^2} \sum_{\ell=0}^{N-1} E\{|\hat{M}(\ell)|^2\}$$

$$= \frac{1}{\sigma^2} \sum_{n=1}^{N} E\{\hat{m}_n^2\}$$

$$= \frac{1}{\sigma^2} E\left\{ \sum_{n=1}^{N} \hat{m}_n^2 \right\}$$

$$= \frac{1}{\sigma^2} E\{\underline{\hat{M}}^T \underline{\hat{M}}\}$$

$$= \frac{1}{\sigma^2} \text{Trace}\{E(\underline{\hat{M}\hat{M}}^T)\}$$

$$= \frac{1}{\sigma^2} \text{Trace}\{E(H\underline{\hat{a}\hat{a}}^T H^T)\}$$

$$= \text{Trace}\{H(H^T H)^{-1} H^T\}$$

$$= K + 1 \qquad (63)$$

also,

$$|G(0)|^2 = \frac{1}{N\sigma^2} E\{|\hat{M}(0)|^2\}$$

$$= \frac{1}{N\sigma^2} E\left\{ \left(\sum_{n=1}^{N} \hat{M}_n \right)^2 \right\}$$

$$= \frac{1}{N} \text{sum}[H(H^T H)^{-1} H^T]$$

$$= 1$$

where for any matrix, A, sum(A) $\triangleq \sum_i \sum_j a_{ij}$. The proof that sum$[H(H^TH)^{-1}H^T]=N$ in showing $|G(0)|^2 = 1$ for white noise input is shown in Appendix A.

Therefore, the equivalent bandwidth (two-sided) of LMSF with white noise sequence is:

$$B_f = (K + 1)\Delta f \text{ (Hertz)}$$

where $\Delta f = \frac{1}{N\Delta t}$ is the DFT frequency sampling interval.

Thus, we obtain the desired result of the equivalent bandwidth as a function of the triplet (K, N, Δt).

$$B_f = \frac{K + 1}{N\Delta t} \text{ (Hertz)} \tag{64}$$

This result is intuitively pleasing since for a fixed sample size, N, increasing K, the order of the fit will pass more noise corresponding to a larger bandwidth.

2.3.4 Unbiased Variance Estimate from Non-Stationary Correlated Data

We assume, as before, that the non-stationary component of the random process is due to the time-varying mean, which can be modeled by a finite-order, algebraic polynomial. The noise process, however, is assumed to be correlated, and an unbiased estimate of the variance from a finite set of observations is sought.

Let $\{X_i\}$; i = 1, 2, ..., N be a sequence of N observations from a random process, the mean and variance of which are:

$$E[\underline{X}] = \underline{M}$$

$$E[(\underline{X} - \underline{M})(\underline{X} - \underline{M})^T] = \sigma^2 R$$

where $\underline{X} = [x_1, x_2, \ldots, x_N]^T$ and R is the normalized covariance matrix. As was shown in the last section, the data can be written in vector form as:

$$\underline{X} = \underline{M} + \underline{W} \tag{65}$$

where

$$E[\underline{W}] = \underline{0}$$

$$E[\underline{W}\underline{W}^T] = \sigma^2 R \tag{66}$$

Assuming that the time-varying mean can be modeled by a polynomial or appropriate set of functionals, then

$$\underline{M} = H\underline{a}$$

where $\underline{a}$ and H are defined as in Equation (45).

The method of least square yields an estimate of $\underline{a}$, the polynomial coefficients, given by:

$$\hat{\underline{a}} = (H^T H)^{-1} H^T \underline{X}$$

It can be seen that:

$$\begin{aligned} E[\hat{\underline{a}}] &= E\{(H^T H)^{-1} H^T \underline{a} + (H^T H)^{-1} H^T W\} \\ &= \underline{a} \end{aligned}$$

and

$$\begin{aligned} E[(\hat{\underline{a}} - \underline{a})(\hat{\underline{a}} - \underline{a})^T] &= (H^T H)^{-1} H^T R H (H^T H)^{-1} \\ &= LRL^T \end{aligned} \tag{67}$$

where

$$L \triangleq (H^T H)^{-1} H^T$$

The error residual is:

$$\underline{e} = H(\underline{a} - \hat{\underline{a}}) + \underline{W} \tag{68}$$

and the standard estimate of the variance gives:

$$\hat{\sigma}_S^2 = \frac{1}{N} \sum_{i=1}^{N} e_i^2$$

$$= \frac{1}{N} \mathrm{Trace}\{\underline{ee}^T\}$$

$$= \frac{1}{N} \mathrm{Trace}\{H(\underline{a} - \hat{\underline{a}})(\underline{a} - \hat{\underline{a}})^T H^T + H(\underline{a} - \hat{\underline{a}})\underline{W}^T + \underline{W}(\underline{a} - \hat{\underline{a}})^T H^T + \underline{WW}^T\} \tag{69}$$

To investigate the bias of the variance estimate, the expected value on both sides of Equation (69) is taken, and yields:

$$E[\hat{\sigma}_S^2] = \frac{\sigma^2}{N} \mathrm{Trace}\{HLRL^T H^T - HLR - RL^T H^T + R\} \tag{70}$$

By noting the relations

$$\mathrm{Trace}\{R\} = N$$

$$\mathrm{Trace}\{HLR\} = \mathrm{Trace}\{RL^T H^T\}$$

$$\mathrm{Trace}\{HLRL^T H^T\} = \mathrm{Trace}\{HLR\}$$

Equation (70) becomes:

$$E[\hat{\sigma}_S^2] = \frac{\sigma^2}{N} [N - \mathrm{Trace}\{HLR\}] \tag{71}$$

We can form the unbiased Kth order fit estimate of the variance from correlated data as:

$$\hat{\sigma}^2_{ubKC} = \nu_K \hat{\sigma}^2_S \tag{72}$$

where

$$\nu_K = \left\{ 1 - \frac{\mathrm{Trace}[H(H^T H)^{-1} H^T R]}{N} \right\}^{-1} \tag{73}$$

Note for uncorrelated noise, $R = I$, and $\mathrm{Trace}[H(H^T H)^{-1} H^T] = K + 1$; therefore Equation (73) reduces to Equation (56) and:

$$\nu_K = \mu_K = \frac{1}{1 - \frac{K + 1}{N}}$$

For zero-order fit,

$$\mathrm{Trace}[H(H^T H)^{-1} H^T R] = \sum_i \sum_j \rho_{ij}$$

Equation (73) reduces to Equation (36) with

$$\nu_0 = \frac{1}{1 - \frac{1}{N_e}}$$

For the general case, let $R = I + R_0$

where R_0 is the matrix, R, with zeros replacing the diagonal elements. The following can then be written as:

$$\mathrm{Trace}\{H(H^T H)^{-1} H^T R\} = \mathrm{Trace}\{I_{K+1}\} + \mathrm{Trace}\{H(H^T H)^{-1} H^T R_0\}$$

$$= K + 1 + \Delta$$

where

$$\Delta \triangleq \mathrm{Trace}\{H(H^T H)^{-1} H^T R_0\}$$

The correction factor can then be written as:

$$\nu_K = \left[1 - \frac{K + 1 + \Delta}{N} \right]^{-1}$$

$$= \left[1 - \frac{K + 1}{N'_e} \right]^{-1} \tag{74}$$

where

$$N'_e \triangleq \frac{N}{1 + \frac{\Delta}{K + 1}} \tag{75}$$

is defined as the effective number of independent samples.

Note that N'_e is determined by the order of the fit and the data correlation characteristic. Also note that in general N'_e is different from N_e, the effective number of independent samples due for a zero-order LMSF. On the other hand, the equivalent bandwidth of the LMSF with uncorrelated data is given by Equation (64) as:

$$B_f = \frac{K + 1}{N\Delta t} \tag{76}$$

and, prior to the LMSF procedure, the correlated data has a bandwidth, B_n, and the effective number of samples, N_e. Furthermore, B_n and N_e are related by:

$$N_e = N\Delta t B_n \tag{77}$$

Using relations Equations (76) and (77) in Equation (74) yields:

$$\nu_K = \left[1 - \frac{K+1}{N\Delta t}\,\Delta t\left(1 + \frac{\Delta}{K+1}\right)\right]^{-1}$$

$$= \left[1 - \frac{B_f}{B_n}\,\varepsilon\right]^{-1} \tag{78}$$

where

$$\varepsilon = \frac{N_e}{N}\left(1 + \frac{\Delta}{K+1}\right)$$

accounts for the correction on B_f due to correlated data.

The remainder of this section investigates the behavior of the equivalent bandwidth correction factor, ε, as shown in Equation (78). For convenience, they are restated as follows:

$$\varepsilon = \frac{N_e}{N}\left(1 + \frac{\Delta}{K+1}\right)$$

where

$$N_e = \frac{N^2}{\sum_i^N \sum_j^N \rho_{ij}}$$

$$\Delta = \mathrm{Trace}[H(H^TH)^{-1}H^TR] - (K+1)$$

and R is the correlation matrix, the ith and jth elements of which are given by ρ_{ij}. Using results developed in Appendix A, the following extreme cases of correlation can be evaluated.

Case (1) - Data uncorrelated: $\rho_{ij} = \delta_{ij}: \forall i,j$

Therefore R = I, and $\mathrm{Trace}\{H(H^TH)^{-1}H^TR\} = K + 1$; furthermore

$\sum_{i}^{N} \sum_{j}^{N} \rho_{ij} = \sum_{i}^{N} \sum_{j}^{N} \delta_{ij} = N$, Hence, $\varepsilon = 1$ is obtained; Namely, no correction is necessary.

Case (2) - Data totally correlated: $\rho_{ij} = 1: \forall i,j$

For this case, $N_e = 1$, and R is a matrix with unity elements. It can then be shown that (using relations obtained in Appendix A):

$$\mathrm{Trace}\{H(H^TH)^{-1}H^TR\} = \mathrm{Trace}\{R\} = N$$

so,

$$\Delta = N - (K + 1),$$

and

$$\varepsilon = \frac{1}{N} \quad 1 + \frac{\Delta}{K + 1}$$

$$= \frac{1}{N} \quad 1 + \frac{N - (K + 1)}{K + 1}$$

$$= \frac{1}{K + 1}$$

Therefore, for any sequence with correlation coefficient ρ, the correction factor, $\varepsilon(\rho)$, must lie between these two extreme cases; i.e.:

$$\frac{1}{K + 1} \leq \varepsilon(\rho) \leq 1 \tag{79}$$

where K is the order of the fit.

Thus, combining Equations (78) and (79), it is concluded that in the presence of correlated data, the effective bandwidth of the LMSF decreases with respect to the case of uncorrelated data.

2.3.5 Equivalent Transfer Function of LMSF in the Presence of Correlated Data

In this section, the expression for the equivalent LMSF transfer function in the presence of correlated data is derived. Our approach to calculate the LMSF equivalent transfer function is as follows:

1. Assume the LMSF behaves as an ordinary filter for a given batch of data sequences,

2. Observe the output spectral response for a given input spectral distribution, and

3. Obtain the magnitude square filter response of the LMSF by taking the ratio of the output spectrum over the input spectrum.

Let $\{x_i\}$, $i = 0, \ldots, N - 1$ be a random input sequence with statistics:

$$E[x_i] = 0 \quad \forall\ i$$

$$E[x_i x_j] = \sigma^2 \rho_{ij} \quad \forall\ i,j \tag{80}$$

where ρ_{ij} is the normalized cross-correlation coefficient.

Let $X(\ell)$ denote the DFT of $\{x_i\}$ so that:

$$X(\ell) = \sum_{n=0}^{N-1} x_n e^{-j \frac{2\pi\ell}{N} n} \tag{81}$$

Then, the averaged power spectrum of the input sequence at frequency ℓ is given by:

$$G(\ell) = E\{|X(\ell)|^2\}$$

$$= E\{X(\ell)X^*(\ell)\}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N} \overline{X_n X_m}\, e^{-j \frac{2\pi\ell}{N}(n - m)}$$

$$= \sigma^2 \sum_{N=0}^{N-1} \sum_{m=0}^{N-1} \rho_{nm} e^{-j \frac{2\pi\ell}{N}(n - m)}$$

$$= \sigma^2(\underline{V}_\ell^* R \underline{V}_\ell) \qquad (82)$$

where $R = [\rho_{ij}]$ is the correlation matrix, the vector $\underline{V}_\ell = \left[1, e^{-j \frac{2\pi\ell}{N}}, \ldots e^{-j \frac{2\pi\ell}{N}(N - 1)}\right]^T$, and $\underline{V}_\ell^*$ is the complex conjugate transpose of $\underline{V}_\ell$. Since the correlation coefficient is symmetric; i.e., $\rho_{mn} = \rho_{nm}$, Equation (82) can be simplified to:

$$G_\alpha(\ell) = \sigma^2 \left\{N + 2 \sum_{K=1}^{N-1} (N - K)\rho(K)\cos\left(\frac{2\pi\ell}{N} K\right)\right\}$$

$$= N\sigma^2 \left\{1 + 2 \sum_{K=1}^{N-1} (1 - \frac{K}{N})\rho(K)\cos\left(\frac{2\pi\ell}{N} K\right)\right\} \qquad (83)$$

where it is assumed that the sequence is wide sense stationary so:

$$\rho_{mn} = \rho(m - n)$$

On the other hand, the output sequence from the LMSF is given by $\{y_i\}$, i = 0, 2, ..., N-1, and, using the vector notation:

$$\underline{y} = H\hat{\underline{a}}$$

where $\hat{\underline{a}}$ is the LMSF solution of the unknown coefficient vector. Equivalently, the output sequence can be written as:

$$y_i = \underline{h}_i^T\hat{\underline{a}}$$

where $\underline{h}_i^T = [1 \ t_i \ \ldots \ t_i^M]$ is the ith row of the matrix H.

Using Equation (67), the following is obtained:

$$E[y_i] = \underline{h}_i^T E[\hat{a}] = 0$$

$$E[y_i y_j] = \underline{h}_i^T E[\hat{\underline{a}}\hat{\underline{a}}^T]\underline{h}_j = \sigma^2 \underline{h}_i^T L R L^T \underline{h}_j \tag{84}$$

where $L = (H^T H)^{-1} H^T$.

Letting $Y(\ell)$ be the DFT of the sequence $\{y_i\}$,

then

$$Y(\ell) = \sum_{n=0}^{N-1} y_n e^{-j\frac{2\pi\ell}{N}n}$$

and the output spectrum is given by the average magnitude square of $Y(\ell)$,

or

$$G_Y(\ell) = E\{|Y(\ell)|^2\}$$

Let $K_{ij} = E[y_i y_j] = \sigma^2 \underline{h}_i^T L R L^T \underline{h}_j$,

then

$$G_Y(\ell) = E\{|Y(\ell)|^2\}$$

$$= E\{Y(\ell)Y^*(\ell)\}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} E[y_n y_m] e^{-j \frac{2\pi\ell}{N} (n - m)}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} K_{nm} e^{-j \frac{2\pi\ell}{N} (n - m)}$$

$$= \underline{V}_\ell^* K \underline{V}_\ell \tag{85}$$

where $\underline{V}_\ell$ is defined as in Equation (82) and K is an NxN symmetric output sequence covariance matrix, the elements of which are defined in Equation (84). Now combining Equations (82) and (85) results in the expression for the LMSF transfer function. Let the magnitude square discrete frequency response of the filter be $|H(\ell)|^2$.

Then

$$|H(\ell)|^2 = \frac{G_Y(\ell)}{G_X(\ell)}$$

$$= \frac{\underline{V}_\ell^* Q \underline{V}_\ell}{\underline{V}_\ell^* R \underline{V}_\ell} \quad ; \ \ell = 0, \ldots, N - 1 \tag{86}$$

where the matrix Q can be written equivalently as

$$Q = \sigma^2 K = HLRL^T H^T$$

It should be mentioned that the frequency response of the LMSF in general depends on the input spectrum or equivalently the input correlation matrix.

2.3.6 Alternate Expression of the Exact Correction Factor

It was shown in Section 2.2, via an intuitive frequency domain argument, that the unbiased correction factor of the variance estimate should be related to the ratio of the output to input bandwidths (Equation (20)). In Section 2.3.4, the exact correction factor for a finite input sequence (Equation (73)) was derived. In Section 2.3.5, the frequency response of the LMSF was calculated. It is intended in this section to show that the exact correction factor, as derived in Section 2.3.4, can be related to the ratio of output to input bandwidths and therefore validate the intuitive results as presented in Section 2.2.

Comparison between Equations (20) and (73) show that the following equivalency must be established:

$$\frac{\mathrm{Trace}[H(H^T H)^{-1} H^T R]}{N} = \frac{B_0}{B_i} \tag{87}$$

where B_0 and B_i are the equivalent bandwidths of the output and input sequences, respectively. Let $\{X_i\}$, 1 = 0, ..., N - 1 be the input sequence, $\{Y_i\}$, i = 0, ..., N - 1 be the LMSF output sequence. Furthermore, let $\{X(\ell)\}$ and $\{Y(\ell)\}$ be the DFT of the input and output sequences, respectively. Then the equivalent input and output bandwidth can be written as:

$$B_i = \frac{\sum_{\ell=0}^{N-1} E[|X(\ell)|^2]}{E[|X(0)|^2]} \Delta f \tag{88}$$

$$B_0 = \frac{\sum_{\ell=0}^{N-1} E[|Y(\ell)|^2]}{E[|Y(0)|^2]} \Delta f \tag{89}$$

But using Equations (81) and (84):

$$E[|X(0)|^2] = E\left[\left(\sum_{n=0}^{N-1} X_n\right)^2\right] = \sigma^2 \mathrm{sum}[R] \tag{90}$$

and

$$E[|Y(0)|^2] = E\left[\left(\sum_{n=0}^{N-1} Y_n\right)^2\right] = \sigma^2 \mathrm{sum}[HLRL^T H^T] \tag{91}$$

where R is the normalized correlation matrix and $L = (H^T H)^{-1} H^T$ is the LMSF operator and sum[R] denotes the element sum of the matrix, R.

Now application of Parseval relation yields:

$$\begin{aligned} \sum_{\ell=0}^{N-1} E[|X(\ell)|^2] &= N\, E\left[\sum_{n=0}^{N-1} X_n^2\right] \\ &= N\, E[\underline{X}^T \underline{X}] \\ &= N\sigma^2 \mathrm{Trace}[R] \end{aligned} \tag{92}$$

and similarly, we have (see Equation (63))

$$\begin{aligned} \sum_{\ell=0}^{N-1} E[|Y(\ell)|^2] &= N\sigma^2 \mathrm{Trace}[HLRL^T H^T] \\ &= N\sigma^2 \mathrm{Trace}[HLR] \end{aligned} \tag{93}$$

Substituting Equations (90) to (93) into Equations (88) and (89) yields:

$$B_i = N \frac{\mathrm{Trace}[R]}{\mathrm{sum}[R]} \Delta f \tag{94}$$

$$B_0 = N \frac{\mathrm{Trace}[HLR]}{\mathrm{sum}[HLRL^T H^T]} \Delta f \tag{95}$$

and their ratio

$$\frac{B_0}{B_i} = \frac{\mathrm{Trace}[H(H^T H)^{-1} H^T R]}{\mathrm{Trace}[R]} \frac{\mathrm{sum}[R]}{\mathrm{sum}[HLRL^T H^T]}$$

$$= \frac{\mathrm{Trace}[H(H^T H)^{-1} H^T R]}{N} \frac{\mathrm{sum}[R]}{\mathrm{sum}[HLRL^T H^T]} \tag{96}$$

since R is a normalized correlation matrix Trace[R] = N. Comparing Equation (96) to Equation (87) shows that the desired result exists if, and only if, the following equality holds:

$$\mathrm{sum}[R] = \mathrm{sum}[HLRL^T H^T] \tag{97}$$

It can be shown that equality in Equation (97) does hold for all values of R (Appendix B). Therefore, the equivalency in Equation (87) is established.

2.4 ERROR ANALYSIS OF THE UNKNOWN MEAN MODEL

Up to this point, only the correction factor for the LMSF acting on the noise process, n(t), has been analyzed and derived. In practice, the choice of fitting function chosen for the LMSF will not match the unknown mean, m(t), exactly. Thus, the residuals of the mismatch between the unknown mean m(t) and the fitting functions will contaminate the variance estimate of the noise process, n(t).

Now to be examined is the error caused by the correction factor, C (Equation (24), which was derived for noise input only), on the variance estimate of the noise process n(t) when the actual input consists of both the unknown mean, m(t), and the noise process, n(t). In addition, the effect on the error by the choice of fitting functions will be commented upon.

Assume that the expected variance of the residual error of an LMSF due to the mismatch in the assumed modeling of the unknown mean process m(t) is given by $\overline{\sigma_\varepsilon^2}$. Also assume the expected variance, σ_m^2, of the residual error of an LMSF due to the noise process, n(t), is related to the actual variance, σ_A^2, of the noise process n(t) by the correction factor, C. Then the error in estimating σ_A^2 by multiplying the expected variance of the residuals of an LMSF due the process m(t) + n(t) by the correction factor, C, is given below.

$$\text{Error}_1 = C \cdot [\overline{\sigma_\varepsilon^2} + \overline{\sigma_m^2}] - \sigma_A^2 \tag{98}$$

or

$$\text{Error}_1 = C\overline{\sigma_\varepsilon^2}$$

The correction factor, C, corrects the zero-mean noise component of the expected variance of the LMSF residual due to the actual input zero-mean noise variance. However, the correction factor, C, will amplify the error component of the expected variance of the LMSF residuals due to the mismatch in the modeling of the unknown input mean. Thus, the above error results.

If no correction was performed, the error in estimating the variance of the input zero-mean noise process n(t) from the expected variance of the LMSF residuals is given by:

$$\text{Error}_2 = \overline{\sigma_\varepsilon^2} + \overline{\sigma_m^2} - \sigma_A^2 \tag{99}$$

or

$$\text{Error}_2 = \overline{\sigma_\varepsilon^2} + \overline{\sigma_m^2} \cdot [1 - C]$$

Let $\overline{\sigma_\epsilon^2}$ be modeled as a factor, α, times $\overline{\sigma_m^2}$

$$\overline{\sigma_\epsilon^2} = \alpha \cdot \overline{\sigma_m^2} \tag{100}$$

A solution can now be conducted for α to determine what percentage $\overline{\sigma_\epsilon^2}/\overline{\sigma_m^2}$ must be for the correction factor, C, to yield a smaller absolute error. This can be done by finding the break even point between the two error expressions.

Squaring and equating Equation (98) and Equation (99) and using Equation (100) yields the following:

$$(\alpha C)^2(\overline{\sigma_m^2})^2 = (1 - C + \alpha)^2(\overline{\sigma_m^2})^2 \tag{101}$$

or

$$(C^2 - 1)\alpha^2 + 2(C - 1)\alpha - (C - 1)^2 = 0$$

The positive solution of Equation (101) is:

$$\alpha = \frac{C - 1}{C + 1}$$

Using the correction factor of Equation (24) yields:

$$\alpha \simeq \left[\frac{K + 1}{N \cdot \Delta t \cdot BW_e(\text{input})}\right] \Bigg/ \left[2 - \frac{K + 1}{N \cdot \Delta t \cdot BW_e(\text{input})}\right] \tag{102}$$

or, assuming $\frac{N + 1}{N \cdot \Delta t \cdot BW_e(\text{input})} \ll 1$,

$$\alpha \simeq \frac{K + 1}{2 \cdot N \cdot \Delta t \cdot BW_e(\text{input})} \tag{103}$$

For the purpose of illustration, assume nominal values of $K = 6$, $\Delta t = 1$, $BW_e(\text{input}) = 1$ and $N = 250$. It yields the following α:

$$\alpha \simeq .014$$

From the above, it is obvious either that the ratio of the input variance of the unknown time varying mean to the input variance of the zero-mean noise must be very small or that the LMSF filter will proportionately remove more of the unknown mean than the noise or both. Fortunately, the near-stationary conditions under which the random bearing error tests are conducted tend to insure both of the above.

As noted earlier for an input white sequence, and a fact which is still valid for reasonable, correlated input sequences, the effect of the LMSF on the noise process n(t) is independent of the form of the fitting functions. However, the form of the fitting function can have a pronounced effect on the removal of the unknown mean process m(t). For example, if the unknown mean was a sinusoid, and the fitting function contained that sinusoid, the expected variance $\overline{\sigma_\varepsilon^2}$ of the LMSF residuals due to the mean would be identically zero. However, if the fitting function did not contain the exact sinusoid there would be an error, and, if the fitting function was a polynomial, the error would be still larger.

If the correction factor, C, is applied to the residuals of the LMSF, then Equation (98) conveys the information that the expected error is always positive and nominally depends only on the expected variance $\overline{\sigma_\varepsilon^2}$ of the LMSF residuals due to the unknown mean process m(t). Therefore, if several forms of the fitting function are used to estimate the variance of the noise process, n(t), after applying the correction factor, C, the smallest estimate is the best.

3.0 COMPUTER-GENERATED DATA

A VAX 11/780 computer was used to compute the theoretical power spectrum and correction factor for the LMSF procedure. The computations were performed using the theoretical equations of Section 2.0, with several selected input parameters. In addition, computer simulations were conducted to verify the theoretical predictions and to compare the old LMSF random error procedure with the new procedure. The following paragraphs describe and illustrate the computer-generated data.

3.1 THEORETICAL DATA

Assuming a white input sequence, Equation (59) can be used to calculate the theoretical discrete power spectrum for a polynomial LMSF filter (i.e., $H_{ij} = t_i^{j-1}$ $i = 1$ to N; $j = 1$ to $K + 1$). Figure 2 illustrates the resulting discrete power spectra, assuming 100 samples (N) sampled at 1 second (Δt) and K = 2, 4, 6 the order of the polynomial LMSF. As can be seen from Figure 2, increasing the order of the polynomial fit proportionately increases the pass band of the polynomial LMSF filter. In all cases, the rolloff is approximately 6 dB/octave at the low-frequency region and the equivalent two-sided bandwidths are exactly given by $(K + 1)/(N\Delta t)$.

Figures 3 and 4 show the theoretical discrete power spectrum for a sin/cos LMSF filter (i.e., $H_{ij} = 1$ for $j = 1$; $H_{ij} = \sin(\frac{\pi}{N} j \cdot t_i)$ for $j = 2, 4, 6, \ldots$; $H_{ij} = \cos(\frac{\pi}{N} (j - 1)t_i)$ for $j = 3, 5, 7, \ldots$; $i = 1$ to N; $J = 1$ to $K + 1$ where K must be even). Similar to Figure 2, the number of samples (N) was held constant at 100, the sampling time (Δt) was 1 second and the order of the fit (K) was set to 4 and 6, respectively. From Figures 3 and 4, the equivalent two-sided bandwidth is obviously given by $(K + 1)/(N\Delta t)$, which is the same as the polynomial LMSF filter. However, the rolloff is infinite as compared to the -6 dB/octave for the polynomial LMSF filter. From the foregoing, it is obvious that although all LMSF filters have identical equivalent bandwidths (input white sequence), the fine structure of their response is highly dependent on the choice of the fitting functions.

Figure 5 shows the theoretical discrete power spectrum (white input sequence) for a polynomial LMSF filter where K, the order of the polynomial fit, is fixed at 6 and N, the number of samples, were set at 50, 75 and 100. The sampling time was 1 second. From Figure 5 it can be seen that the equivalent bandwidth of the discrete power spectrum is inversely proportional to the number of samples, N.

A parameter of practical interest is the ratio (r) of the equivalent bandwidth of the LMSF filter to the input equivalent bandwidth (BW_e) defined below.

$$r = \frac{K + 1}{N \cdot \Delta t \cdot BW_e} \tag{104}$$

The input bandwidth BW_e can often be accurately estimated. Therefore, r can be used to relate the parameters of the LMSF filter (K, N) to the input bandwidth (BW_e). In Equation (24), the approximate correction factor between the sum of squared residuals of the LMSF and input variance, can be expressed in terms of r.

$$\text{Approximate Correction Factor} = \frac{1}{1 - r} \tag{105}$$

Figures 6, 7 and 8 illustrate the theoretical percentage error in the standard correction factor (Equation (10)) for white input sequence and the approximate correction factor (Equation (24)) for correlated input sequence versus r for K, the order of the polynomial LMSF equal to 0, 3 and 6, respectively. The number of samples (N) was held constant at 100 and the sampling time was 1 second. The correlated noise is modeled as the sampled output of a simple one-pole, low-pass filter to a unit variance white noise input. The correlation function for the correlated input sequence is given by:

$$R_{ij} = \alpha^{|i-j|}, \text{ where } \alpha \text{ is the filter coefficient} \tag{106}$$

and the two-sided equivalent bandwidth is given by (see Appendix C):

$$BW_e = (1 - \alpha)/(1 + \alpha) \tag{107}$$

Equation (73) is used to calculate the exact correction factor.

Figure 6 confirms the earlier stated fact that the approximate correction factor is exact for K = 0. Although Figures 7 and 8 show that the approximate correction factor for $r > .5$ has a greater percentage error than the standard correction factor, the approximate correction factor is significantly better than the standard correction factor in the useful region where $r \leq .25$. A good rule of thumb would be to choose the parameter (N, Δt, K) of the LMSF such that r remained less than .25 for a given correlated input noise sequence.

Since the LMSF filter is dependent on the input data, the power transfer function will also be a function of the input data. Therefore, it is obvious that the power transfer function of the LMSF filter will also be dependent on the correlation or equivalent bandwidth of the input sequence. However, if the equivalent bandwidth of the LMSF filter is small compared to the equivalent bandwidth of the input sequence, the pass band and initial rolloff of the LMSF filter power transfer function will be only slightly affected. Figures 9 and 10 show the comparison between the power transfer function for an input white sequence and an input correlated noise sequence ($r = .25$). Figure 9 is for a 3rd order polynomial fit and Figure 10 is for a 6th order polynomial fit. In both cases, the number of samples was 100 and the sampling time was 1 second. The correlated input sequence was the output of a simple, low-pass filter to a white noise input. From examining Figures 9 and 10 it is obvious that the power transfer functions for the input white and correlated input sequences are identical until -15 dB, where the correlated power transfer function flattens out. As expected, there is no change in the power transfer function of the zero order polynomial LMSF filter with correlated input data. More surprising is that the power transfer function for the sin/cos LMSF filter is not a function of the correlation of the input sequence for any order fit. In summary, the power transfer function is dependent on the correlation of the input noise sequence and the fitting function of the LMSF.

3.2 SIMULATION DATA

In order to verify the theoretical prediction, Monte-Carlo simulations were performed on selected theoretical predictions.

Figures 11 and 12 compare the simulated and theoretical power transfer function with a white noise input for a 3rd and 6th order polynomial LMSF filter, respectively. The simulated power transfer function was calculated by taking the FFTs of the input psuedo-random Gaussian sequence, and the corresponding output sequence. The output power-transfer function consists of taking 10 log of the ratio of the output to the input magnitude squared of the FFT data. The simulation was repeated and averaged 1000 times. Figures 11 and 12 indicate a reasonable agreement between the simulation and the theoretical results.

Figure 13 compares the simulated and theoretical percentage error in the approximate correction factor (Equation (105)) as a function of r, the ratio of the equivalent bandwidth of the LMSF filter and the equivalent bandwidth of the correlated input sequence. The simulation is for a 6th order polynomial fit consisting of 128 samples sampled at a 1-second rate. The correlated input sequence was formed by passing psuedo-Gaussian random number (X_i) through a simple digital filter, given below:

$$Y_i = \alpha Y_{i-1} + (1 - \alpha)X_i \tag{108}$$

where α is a filter parameter.

In the limit (infinite samples), the equivalent input bandwidth is given by:

$$BW_{input} = (1 - \alpha)/(1 + \alpha) \tag{109}$$

The expected value of the simulated correction factor was calculated by averaging the squared residuals of a 6th order polynomial LMSF 1200 times. The actual correction factor was determined by measuring the input and output

variances for 153,600 samples and then dividing the input variance by the output variance. As can be seen from Figure 13, the simulated data agrees quite well with the theoretical predictions.

4.0 NEW RANDOM ERROR CALCULATION TECHNIQUE

The following outline summarizes the new technique in tracker random error calculation.

1. Perform, in parallel, both a polynomial and sin/cos LMSF on the bearing error data in order to remove the unknown mean.

2. In order to obtain a good statistical average, choose N, the number of samples, to be as large as possible under the constraint that the indicated SNR remains relatively constant over the entire interval. The above should ensure that the tracker bearing data is a valid function of the averaged SNR. In addition, to ensure numerical and statistical validity, constrain N to fall between $60 \leq N \leq 600$. Δt, the sampling rate, is set by the data rate to be 1 second.

3. Maintain the ratio, r, of the equivalent LMSF bandwidth to the equivalent input bandwidth at a value less than .25 by selecting the order of the LMSF using the following equation:

$$K = \text{Nearest Integer}[r \cdot BW_e \cdot N \cdot \Delta t - 1] \tag{110}$$

If $K < 0$, then $K = 0$
If $K > 6$, then $K = 6$

where $r = .2$.

BW_e = is the estimated two-sided bandwidth of the input data
N = the number of samples
Δt = 1
K = the order of the LMSF.

For the sin/cos LMSF, K is rounded to the nearest even integer (0, 2, 4, 6).

4. In order to remove the bias in the variance measurement resulting from the LMSF, multiply the squared residuals of the LMSF by the following correction factor:

$$\text{correction factor} = \frac{1}{1 - \left(\frac{K + 1}{N \cdot \Delta t \cdot BW_e}\right)} \qquad (111)$$

the equivalent two-sided input bandwidth can be estimated from the averaged SNR.

5. Select the smaller of the two variance estimates resulting from the polynomial and sin/cos LMSFs.

6. Associate the resulting bearing variance estimate with the linear averaged SNR over the interval.

As a final test, a simulation was performed comparing the new random error technique to the previous random error technique (para. 1.1), using a model of a modern sonar broadband tracker to generate bearing data. The simulation was performed assuming that the unknown mean was zero. Therefore, the simulation tested only the ability to correctly estimate the input variance from the sum of squared residuals of the LMSF. However, in general, the higher the order of the LMSF the better it can remove the unknown mean and thus yield an accurate random error measurement. The simulation was similar to the one described in paragraph 3.2 except that a model of the actual broadband tracking loop was used instead of the simple low-pass filter given in Equation (108). The measured equivalent bandwidth and variance of the modeled broadband tracker was in excellent agreement with the theoretical prediction, implying a good model. Figure 14 plots the order of the polynomial LMSF selected, as well as the percentage error for both the new and old random error procedures. The simulation was performed using a sample size of 128 and only polynomial fitting functions. From Figure 14 it can be seen that the percentage error using the new technique is always better than the percentage

error in the old technique, and almost always less than a few percent. In addition, except for a minor exception at -5 dB, the order of the polynomial fit for the new technique is always greater than or equal to the old technique. In summary, the new random error procedure should represent a significant improvement over the old random error technique.

5.0 SUMMARY AND CONCLUSIONS

1. A Least Mean Square Fit (LMSF) procedure is used to remove an unwanted and unknown time-varying mean from recorded noisy and biased bearing error data. The residual resulting from the LMSF is used to estimate the bearing variance.

2. The effect of the LMSF on the time-varying mean can be analyzed separately from the analysis of the effect of the LMSF on the noisy unbiased bearing error data; provided that the zero-mean noise data is independent of the time-varying mean.

3. The LMSF can be modeled and analyzed as a low-pass filter acting on a white or correlated noise input sequence.

4. The calculation of variance based on LMSF residuals is known to be biased and an appropriate correction factor is needed. For the case of uncorrelated data, the correction factor is known and is determined by the order of the fit, the number of samples and the sampling interval. It was shown in this study that the correction factor can be equated to the equivalent bandwidth of the LMSF when treated and analyzed as a filter. This study extends the classical correction factor to include the case of correlated data. The exact correction factor has been derived and is shown to be related to the covariance matrix of the input sequence (Equation (73)).

5. An excellent approximation, which depends on the estimate of the input equivalent two-sided bandwidth, has been derived which corrects the sum of squared residuals of the LMSF to an unbiased estimate of the desired input variance.

6. Based on the theoretical analysis of the LMSF filter and the derived correction factor, a new technique has been developed and applied to estimate tracker random error. This new procedure shows substantial theoretical improvement over the previous procedure.

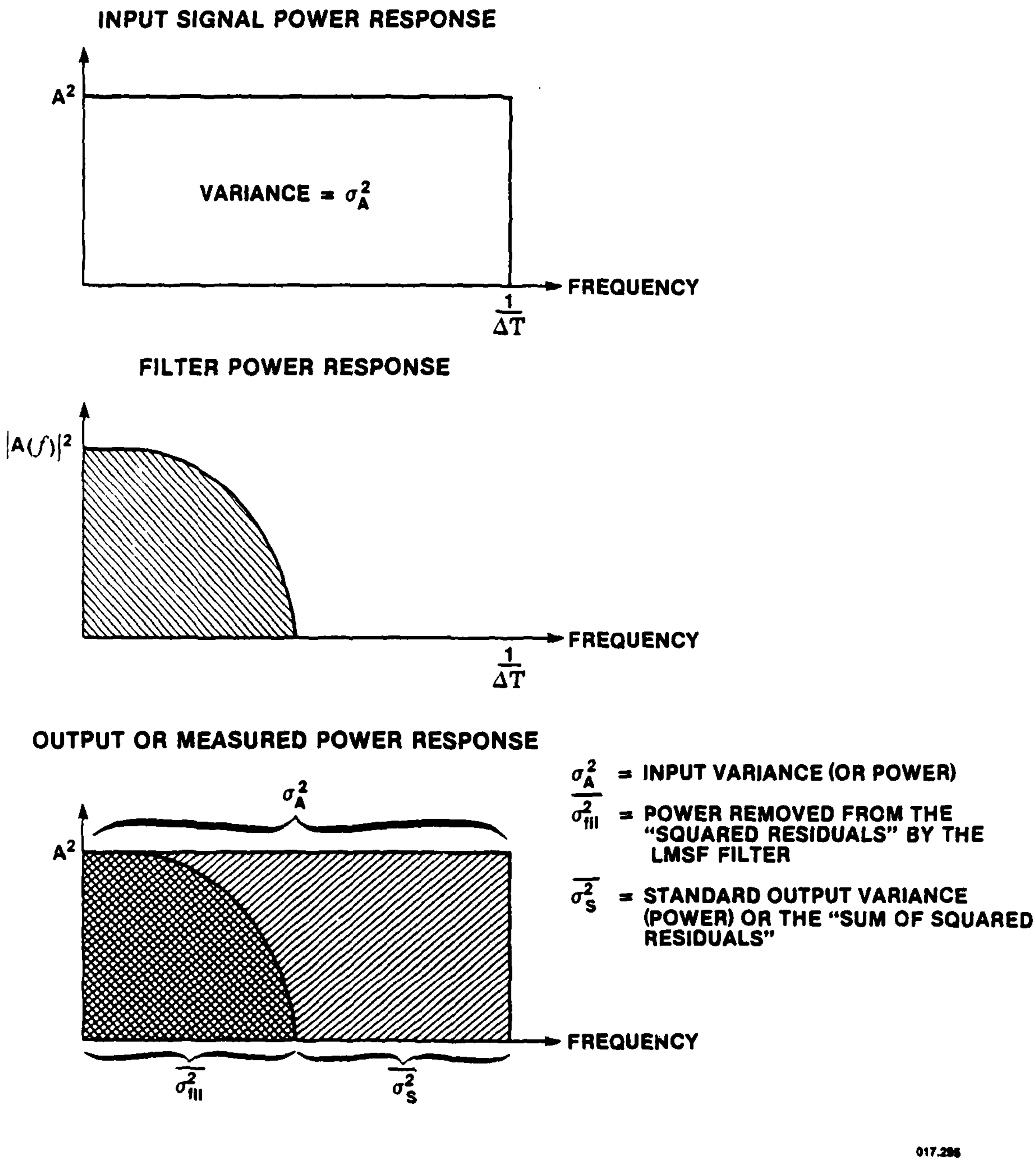

Figure 1. Illustration of Input Variance to Output Variance Relationship

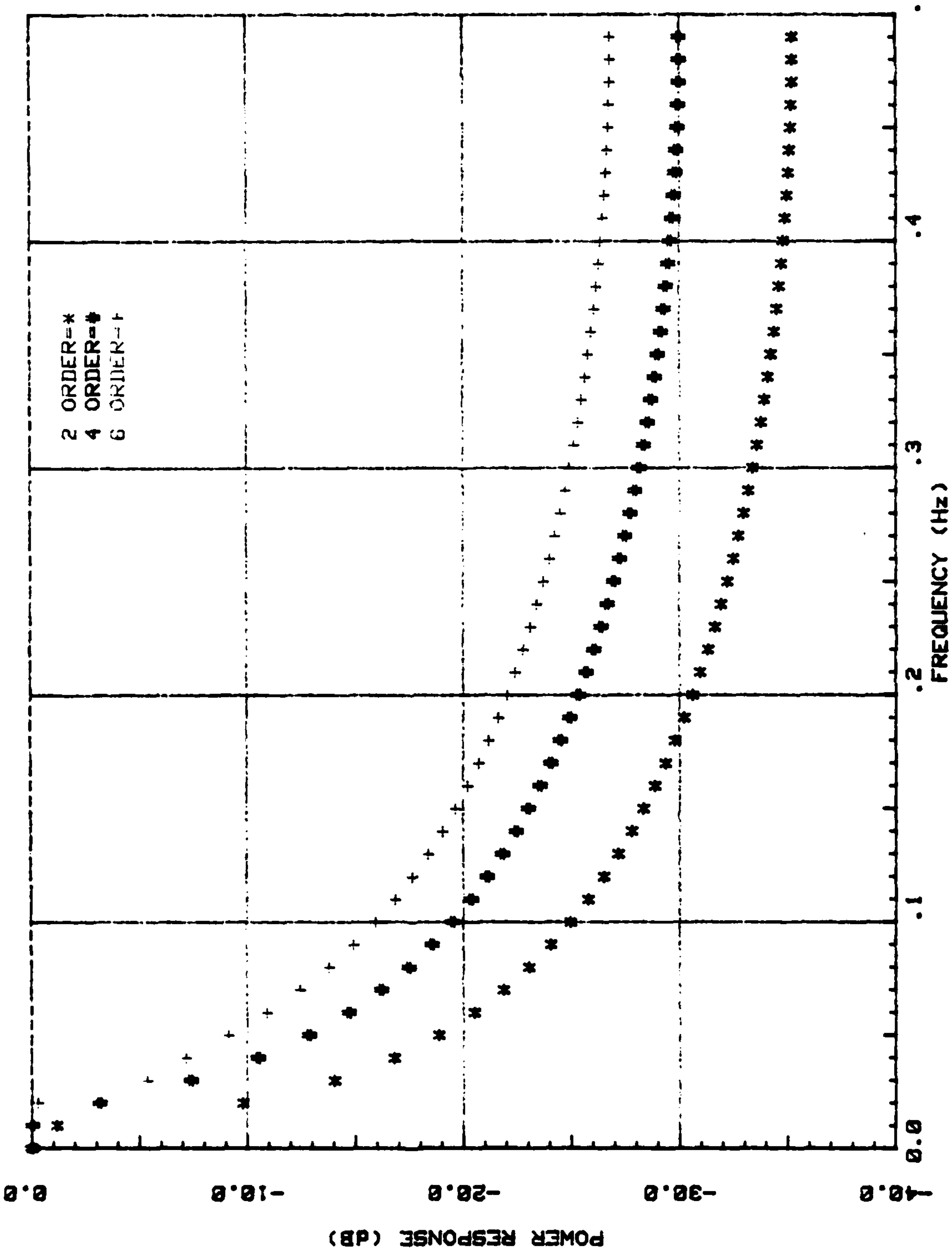

FIGURE 2 LMSF POWER VS FREQUENCY
100 SAMPLES

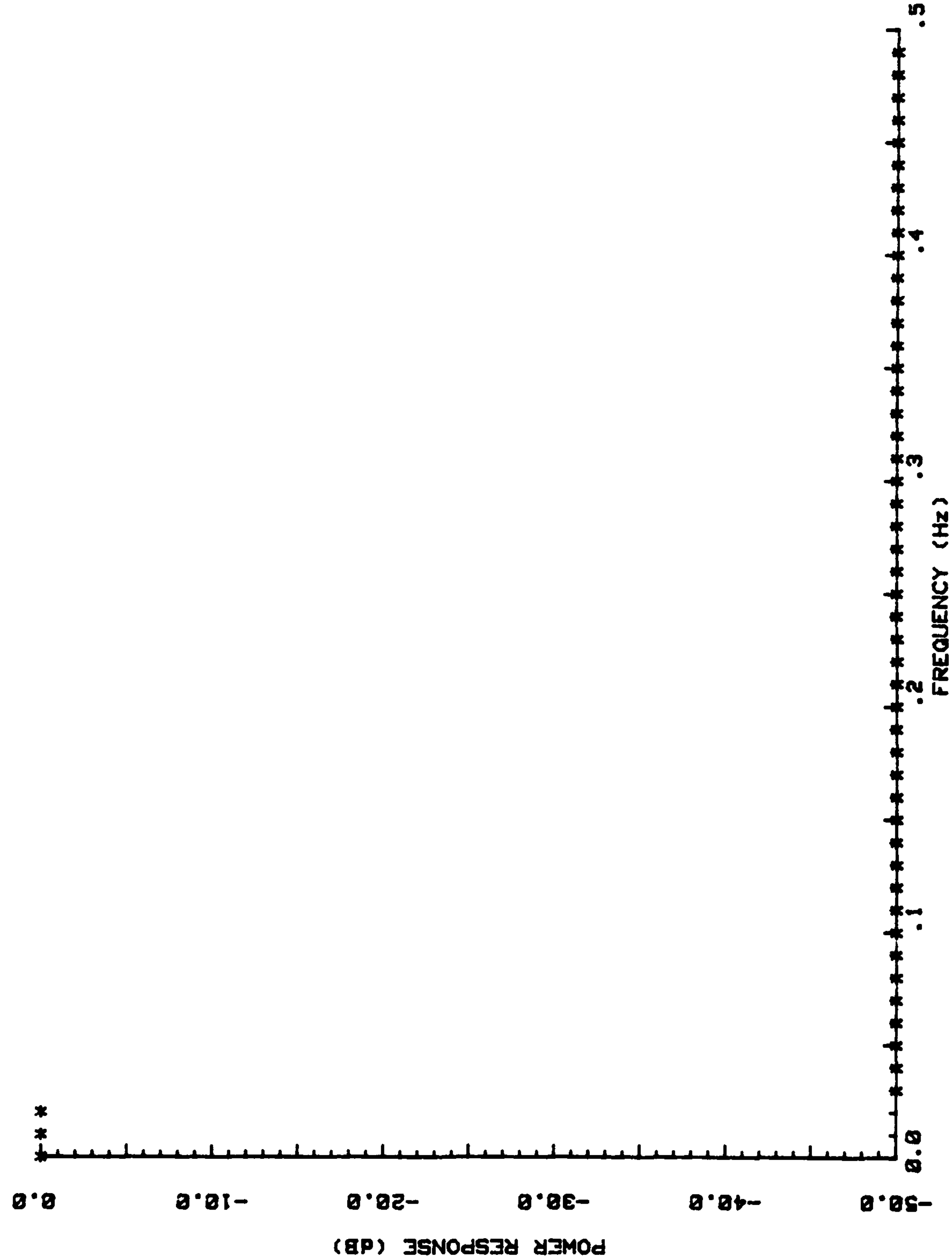

FIGURE 3 LMSF POWER VS FREQUENCY
4th ORDER SIN/COS FIT 100 SAMPLES

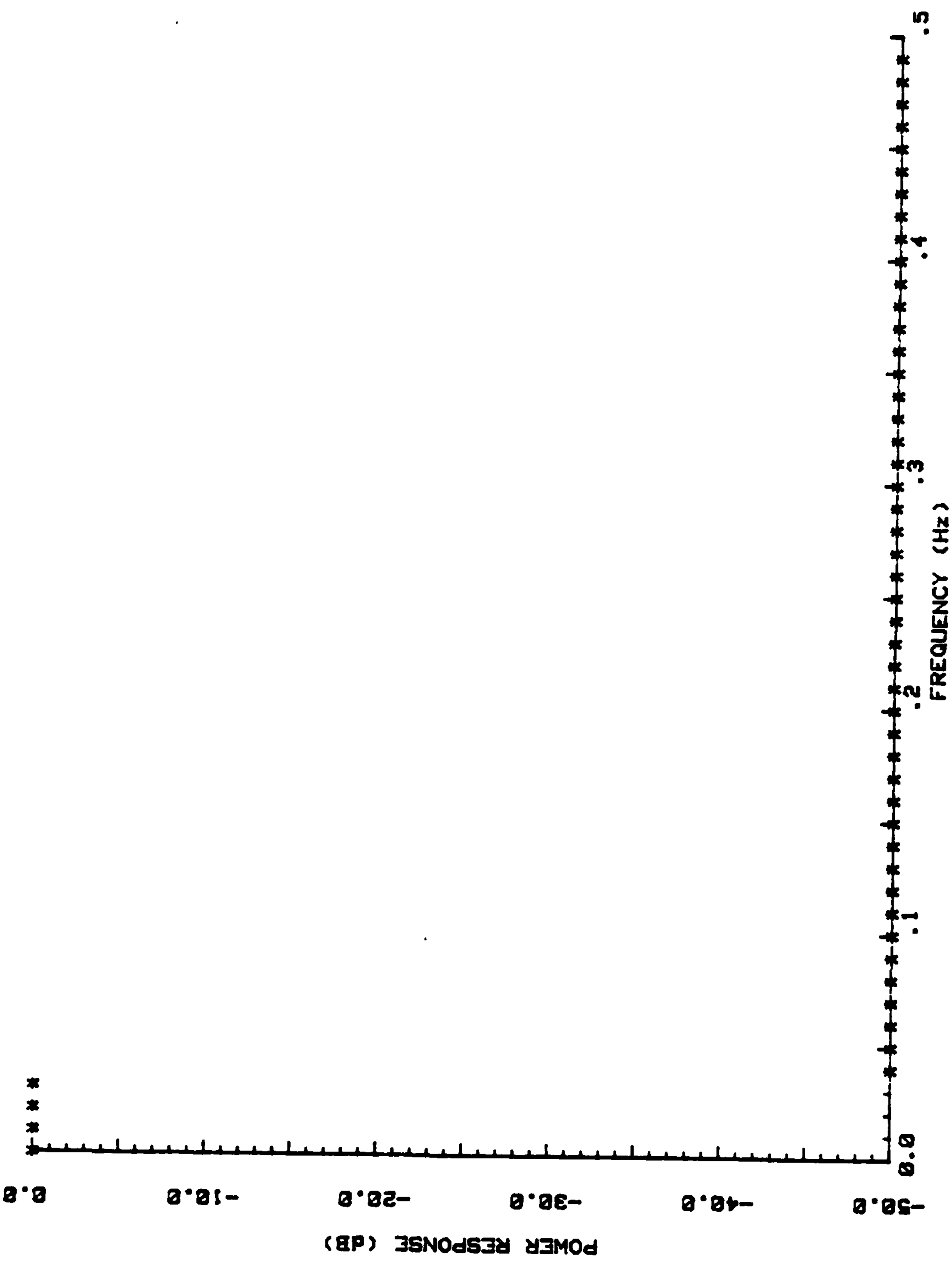

FIGURE 4 LMSF POWER VS FREQUENCY
6th ORDER SIN/COS FIT 100 SAMPLES

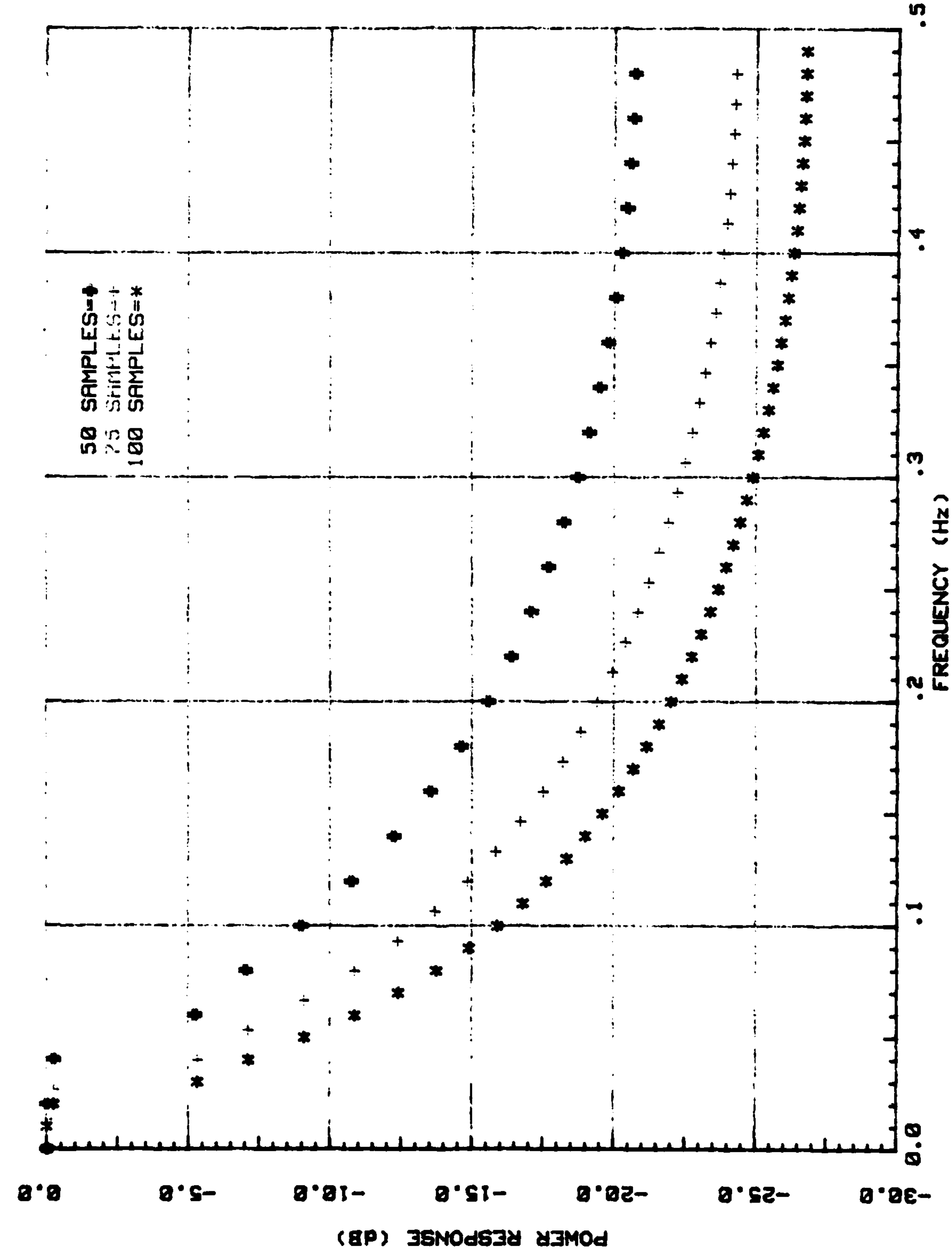

FIGURE 5 LMSF POWER VS FREQUENCY
6th ORDER POLYNOMIAL FIT

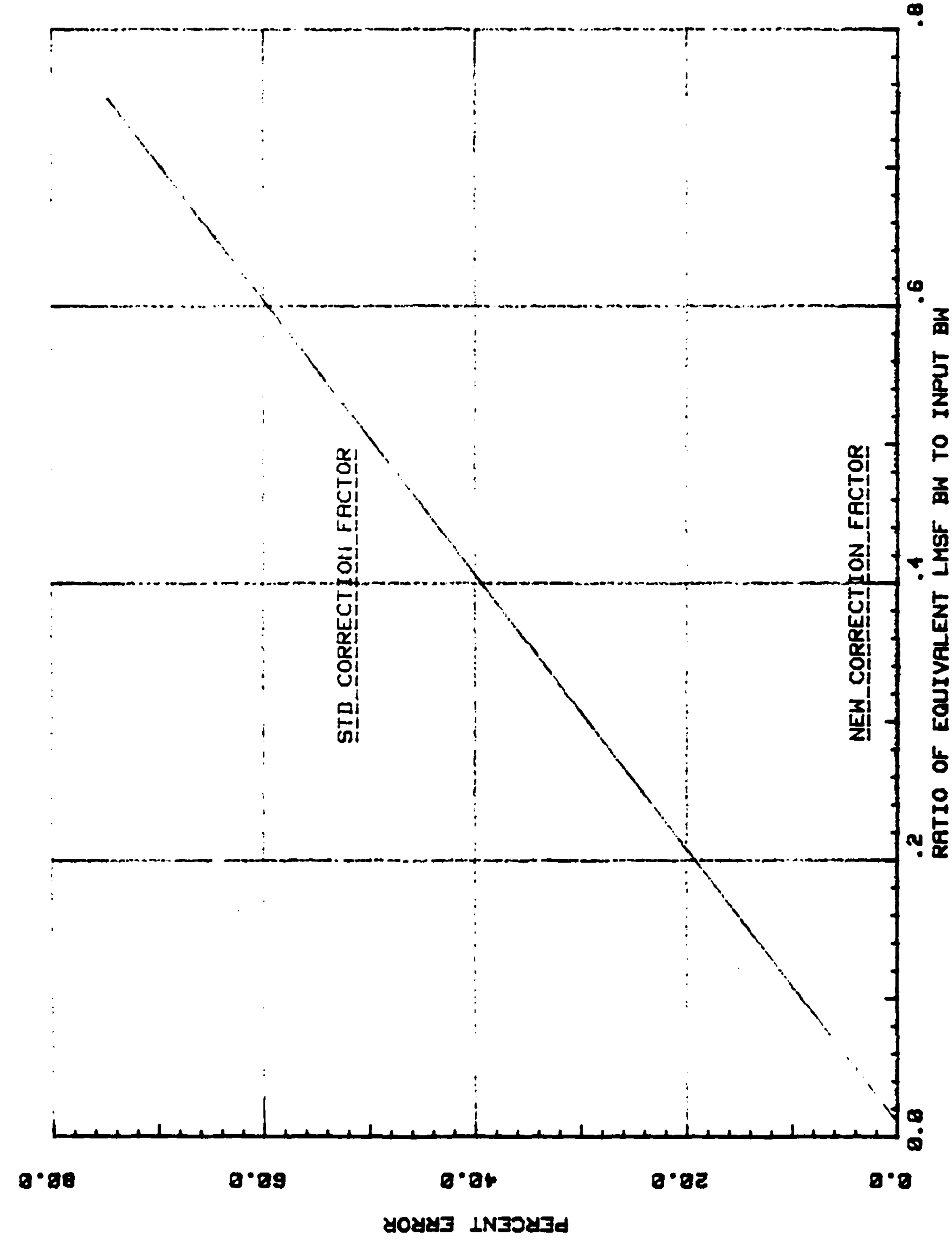

FIGURE 6 % ERROR OF STD & NEW CORRECTION FACTOR VS RATIO OF BANDWIDTHS
0th ORDER POLYNOMIAL FIT 100 SAMPLES

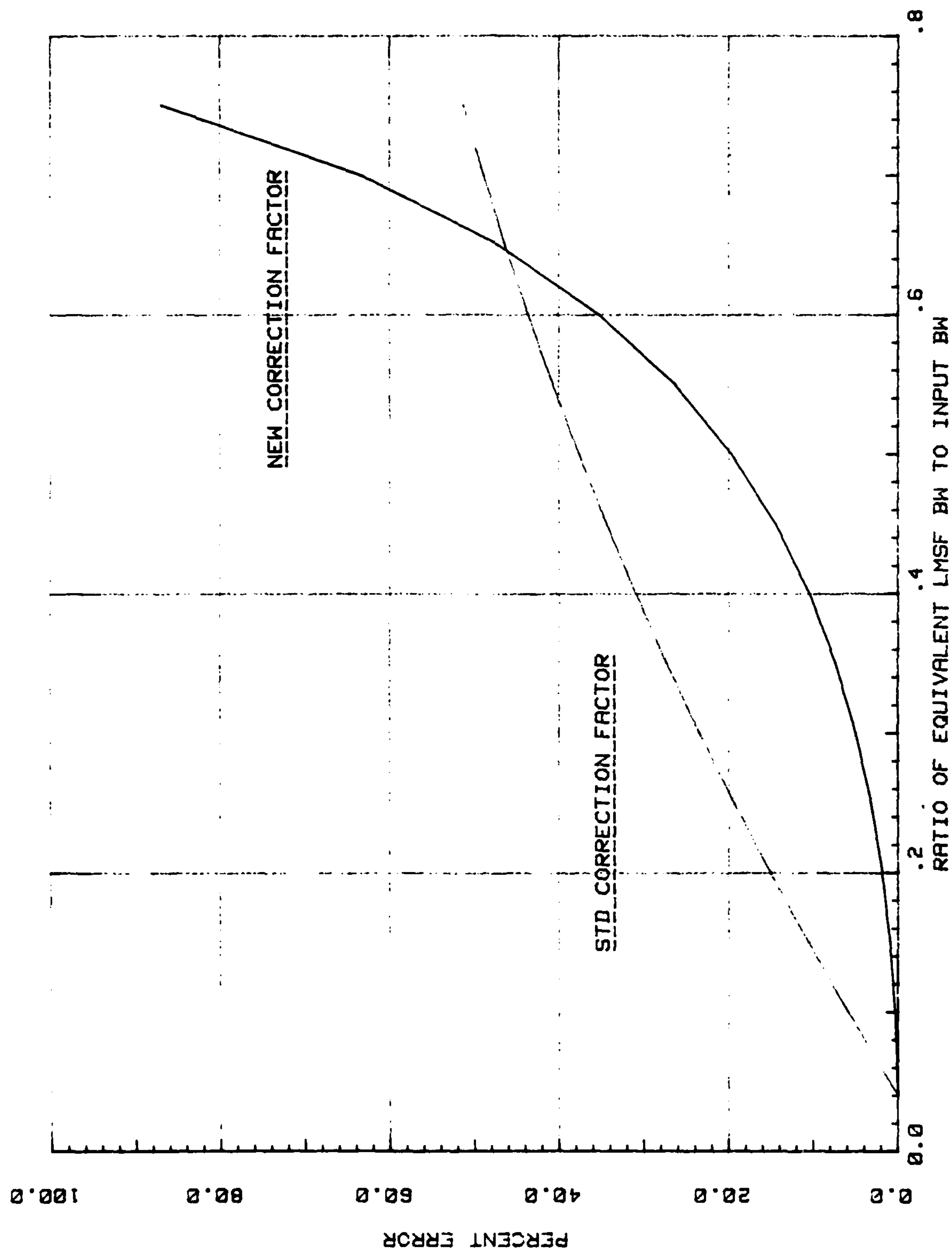

FIGURE 7 % ERROR OF STD & NEW CORRECTION FACTOR VS RATIO OF BANDWIDTHS
3rd ORDER POLYNOMIAL FIT 100 SAMPLES

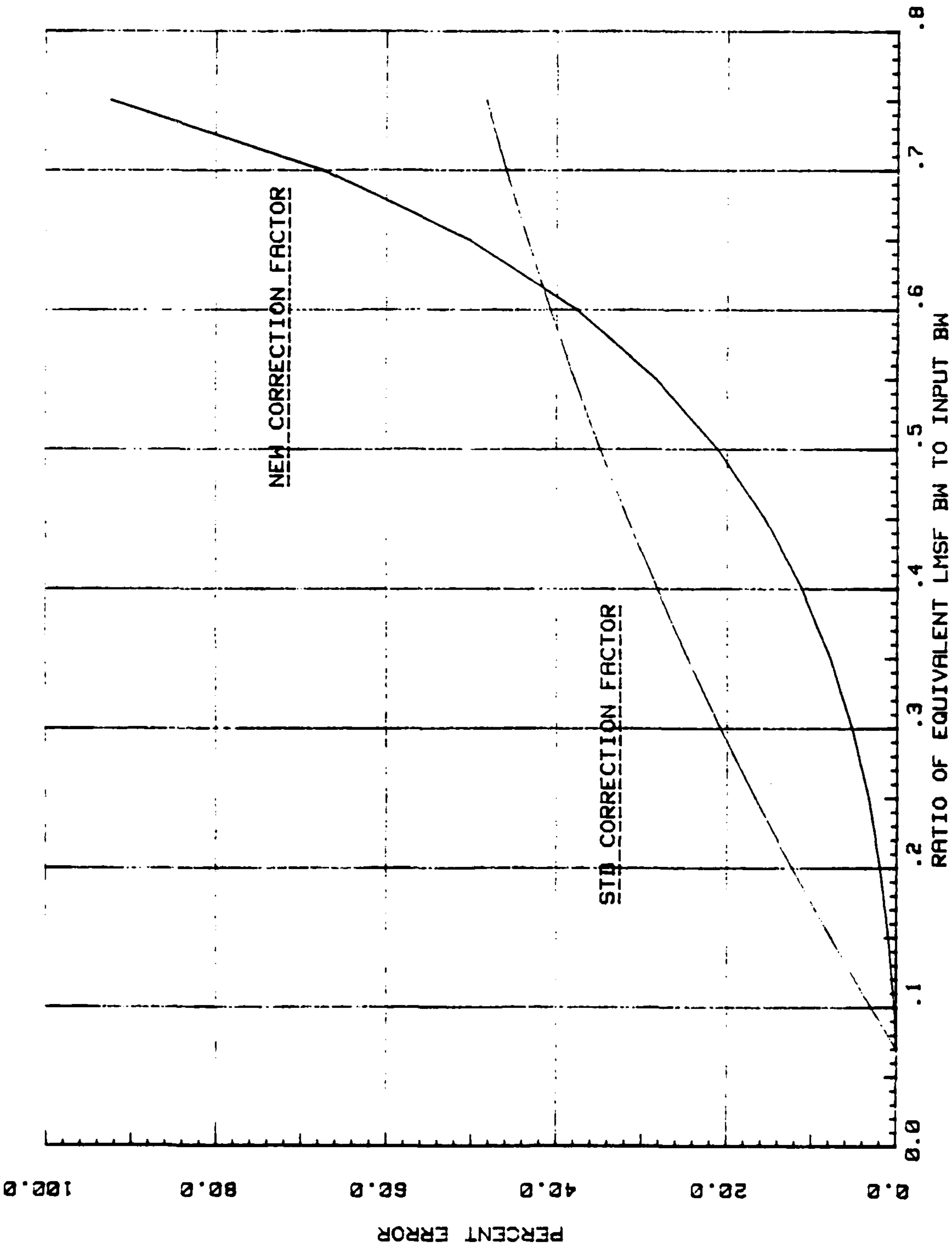

FIGURE 8 % ERROR OF STD & NEW CORRECTION FACTOR VS RATIO OF BANDWIDTHS
6th ORDER POLYNOMIAL FIT 100 SAMPLES

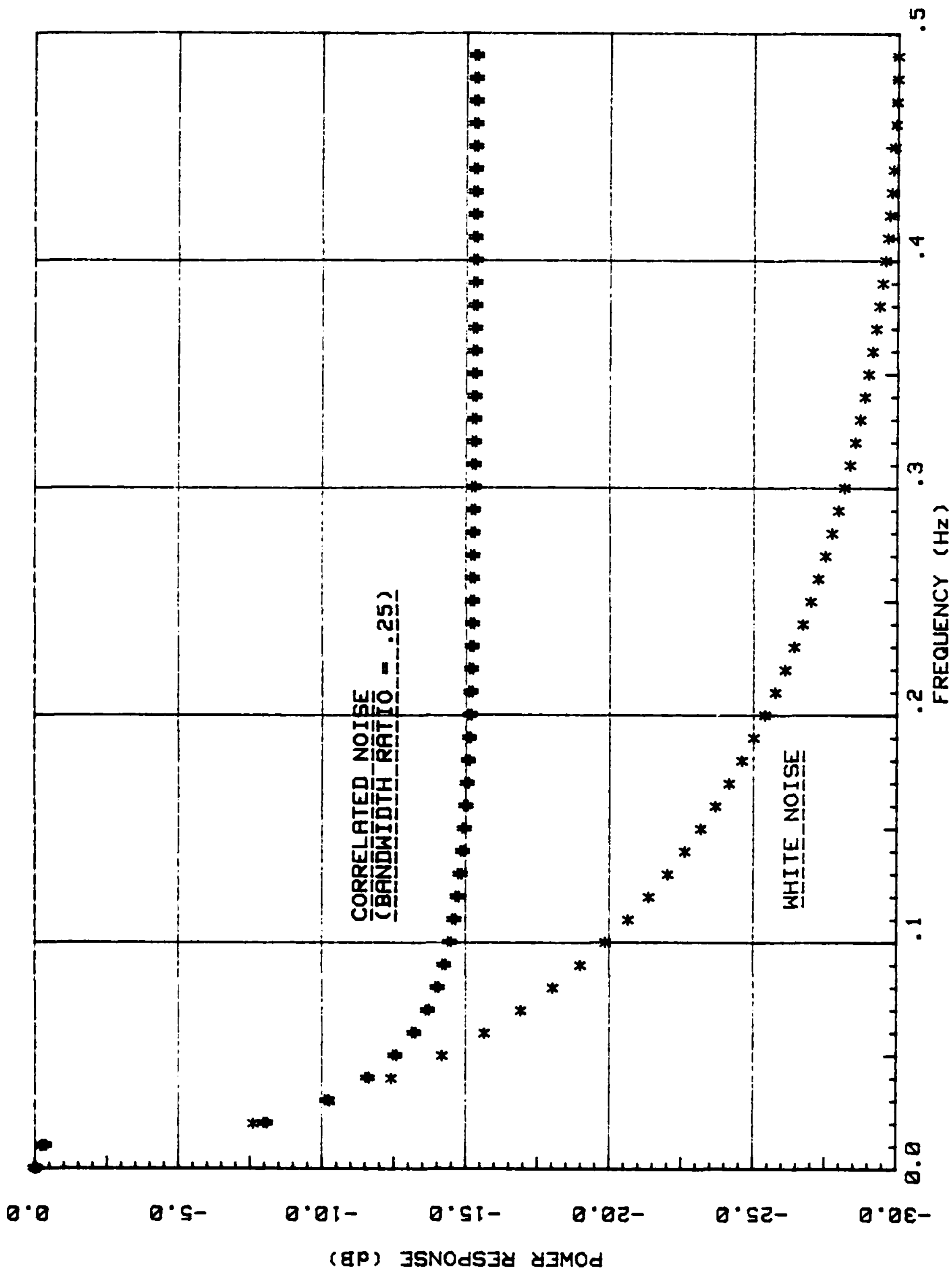

FIGURE 9 LMSF POWER VS FREQUENCY
3rd ORDER POLYNOMIAL FIT 100 SAMPLES

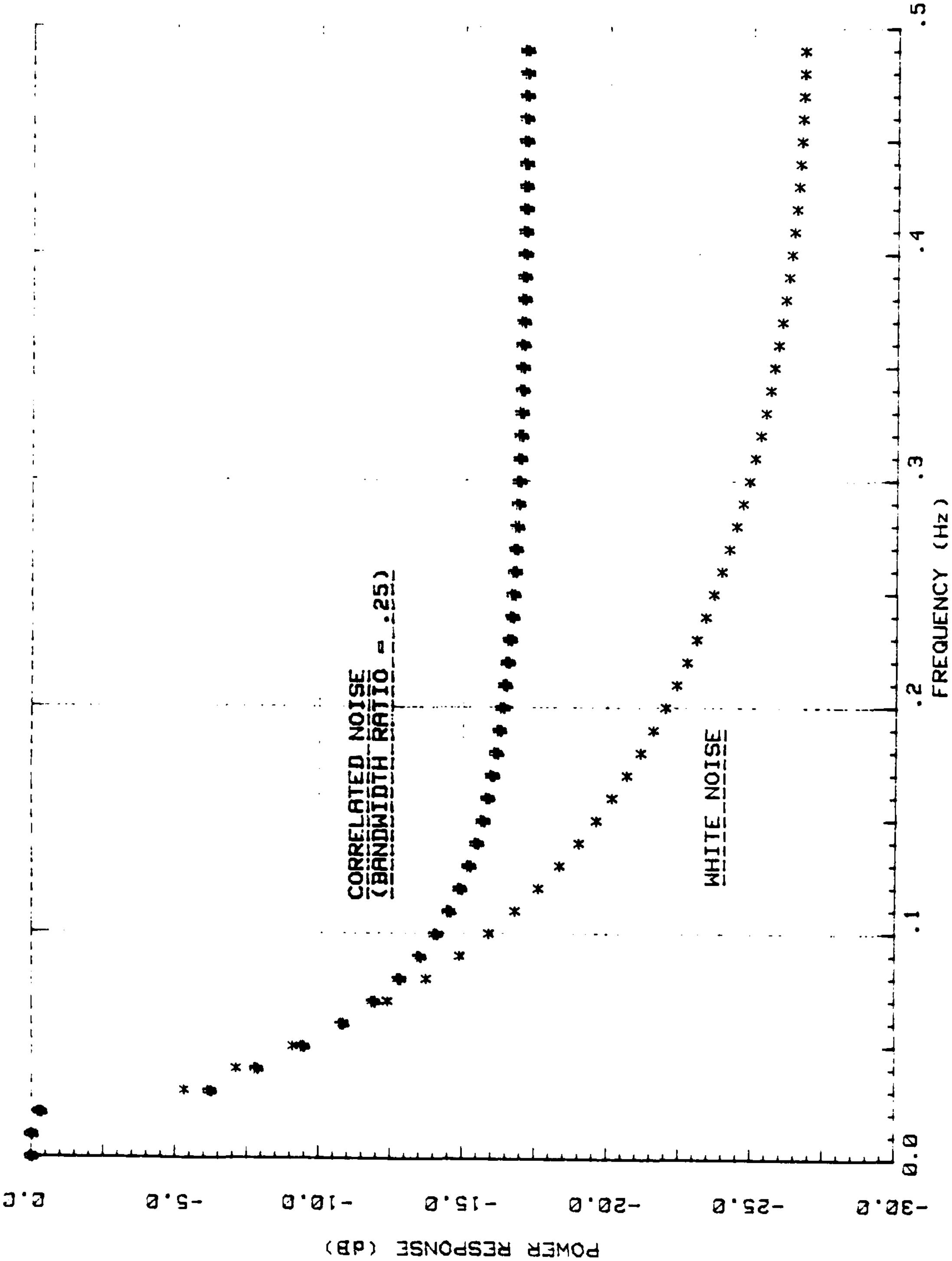

FIGURE 10 LSMF POWER VS FREQUENCY
6th ORDER POLYNOMIAL FIT 100 SAMPLES

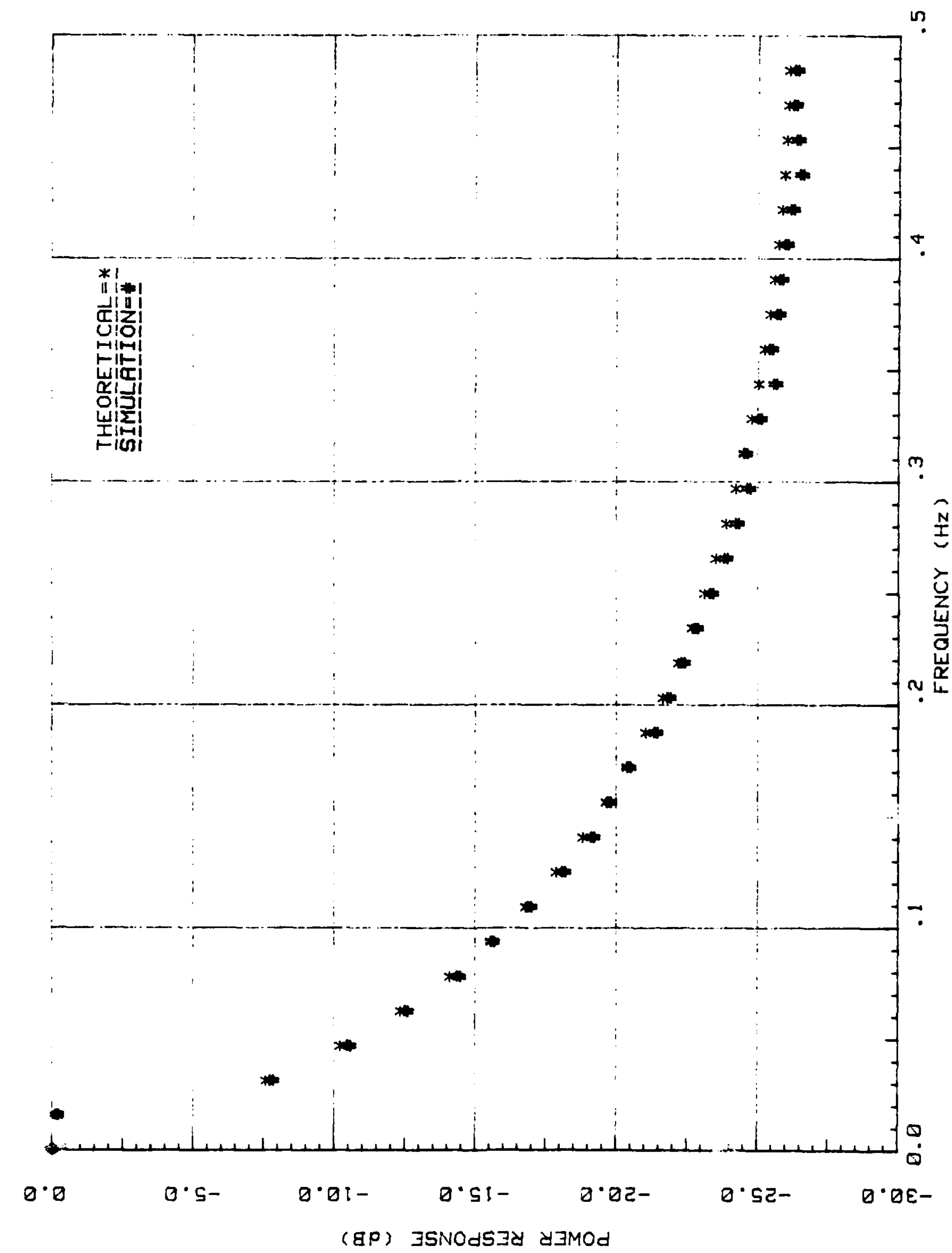

FIGURE 11   LSMF POWER VS FREQUENCY
3rd ORDER POLYNOMIAL FIT 64 SAMPLES

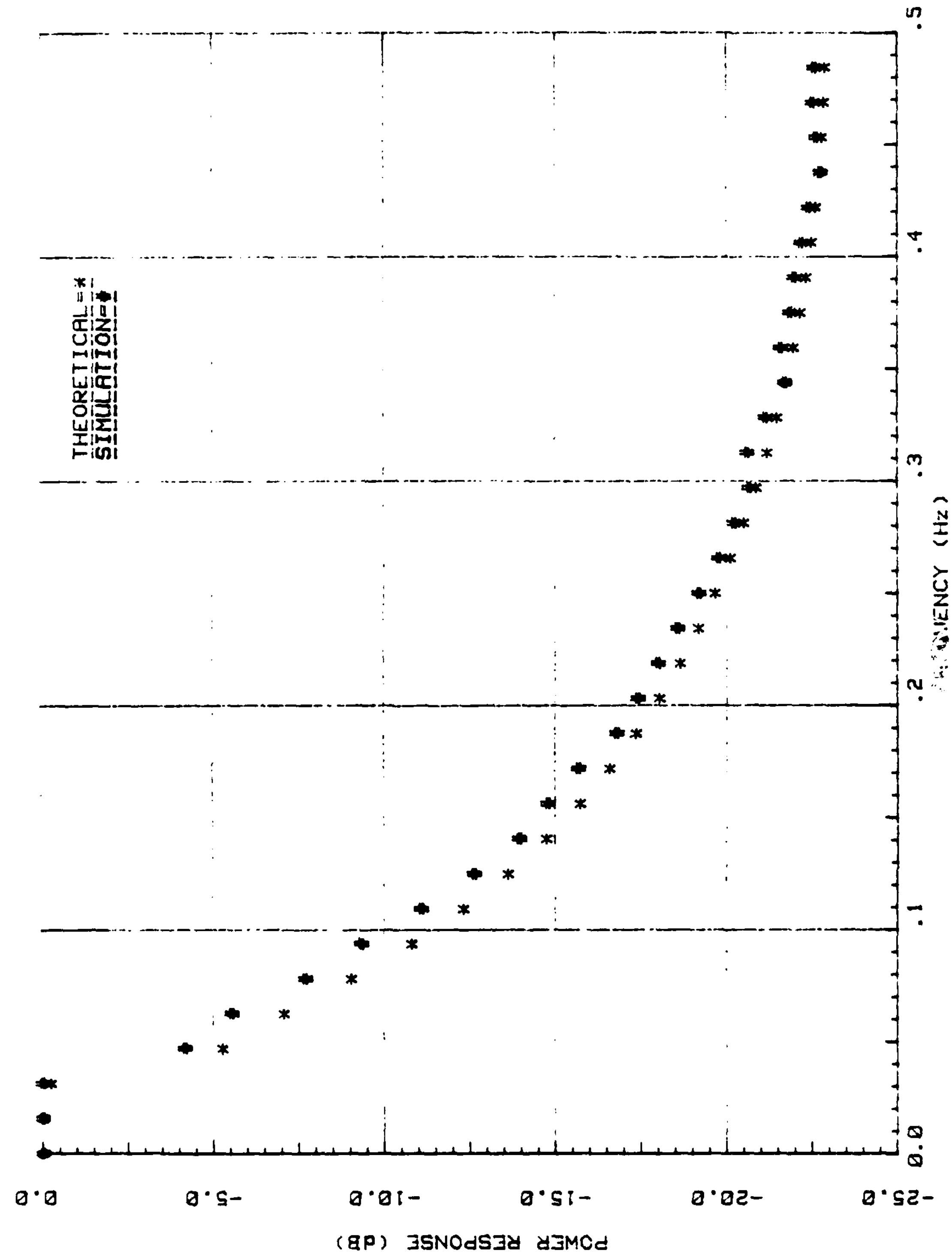

FIGURE 12 LSMF POWER VS FREQUENCY
6th ORDER POLYNOMIAL FIT 64 SAMPLES

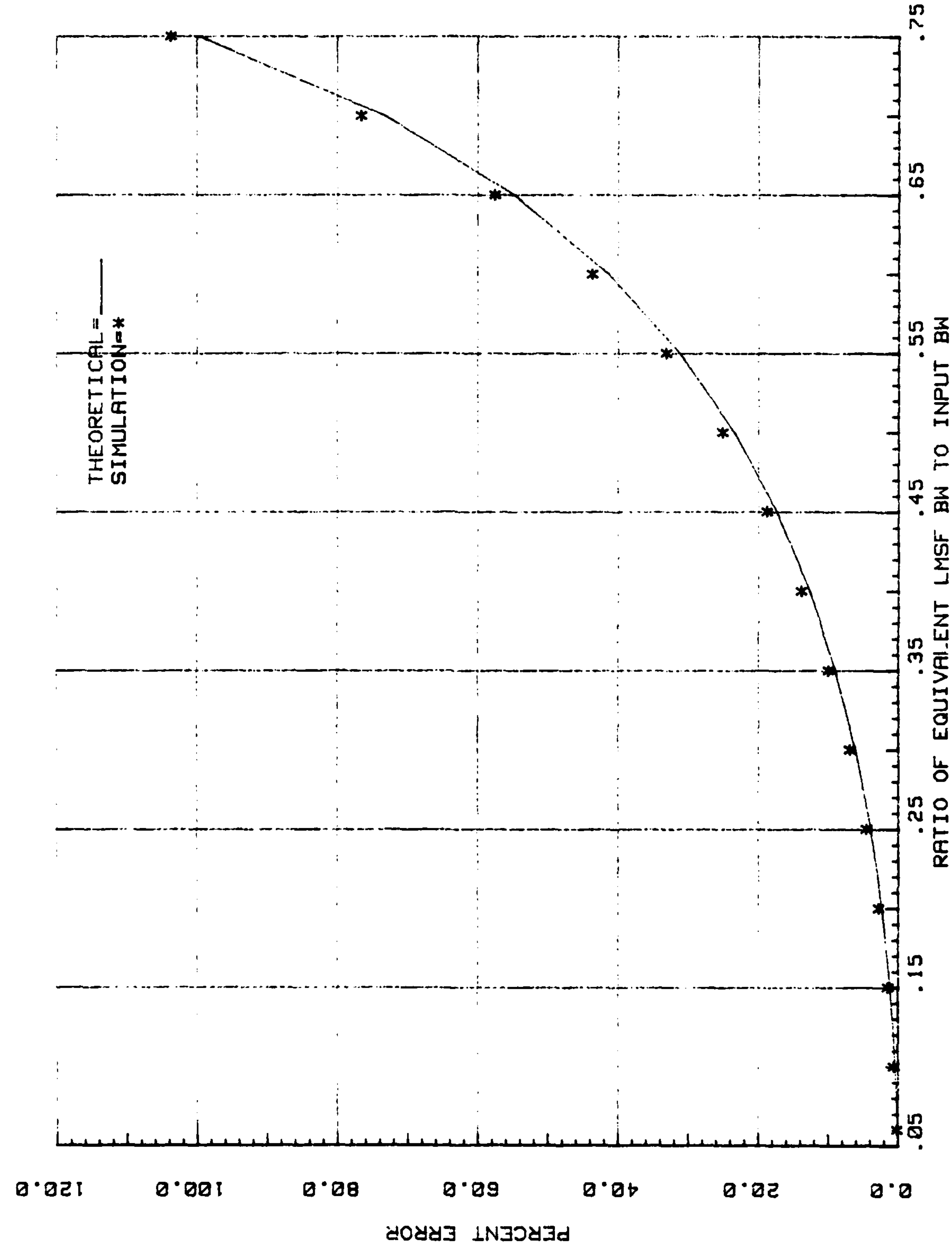

FIGURE 13 % ERROR OF NEW CORRECTION FACTOR VS RATIO OF BANDWIDTHS
6th ORDER POLYNOMNIAL FIT 128 SAMPLES

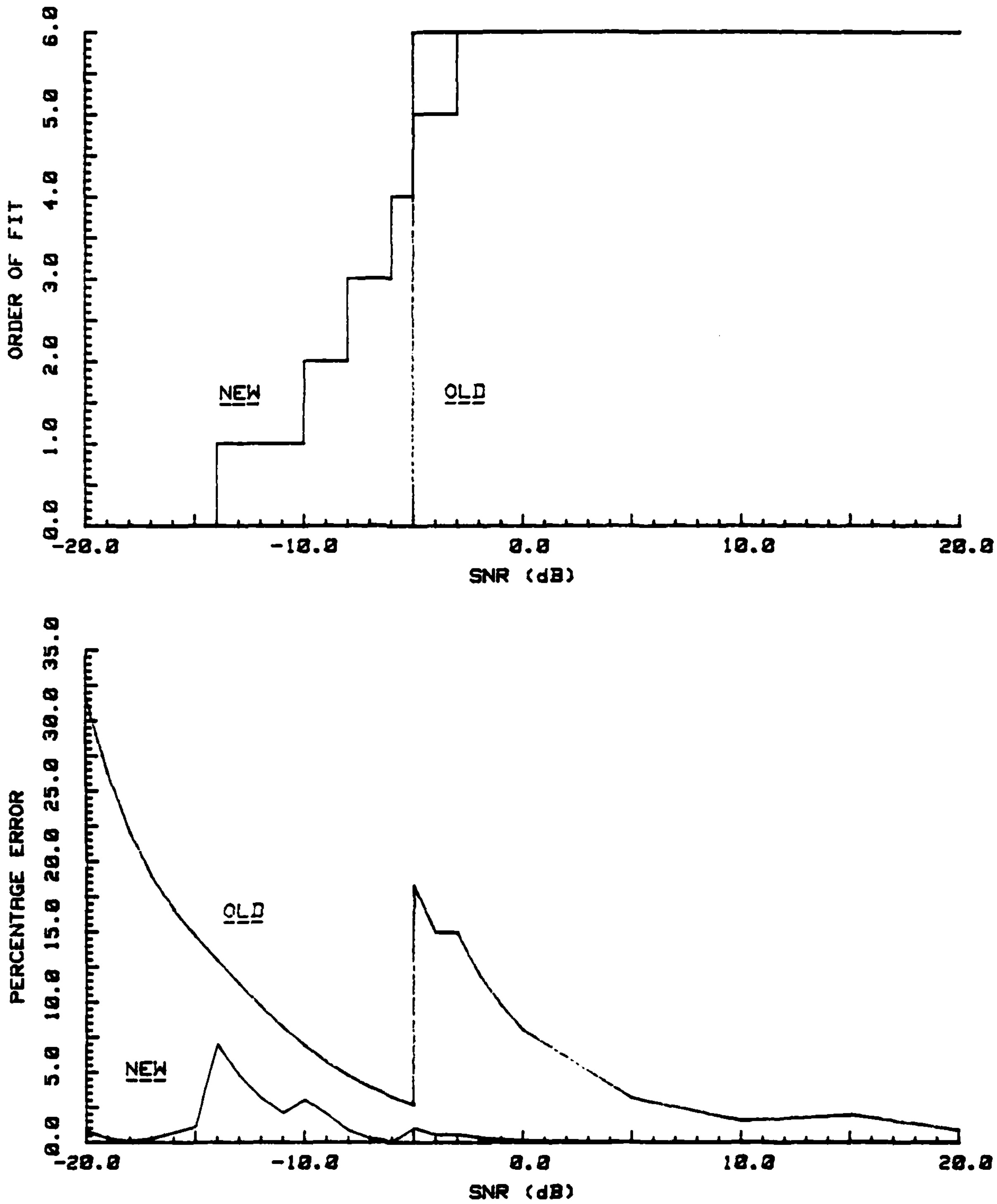

FIGURE 14 COMPARISON OF NEW VS OLD RANDOM ERROR TECHNIQUES

APPENDIX A
PROOF OF $\mathrm{sum}[H(H^TH)^{-1}H^T] = N$

Recall that

$$H = \begin{bmatrix} 1 & t_1 & \cdots & t_1^K \\ 1 & t_2 & \cdots & t_2^K \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & t_N & \cdots & t_N^K \end{bmatrix} \tag{A-1}$$

is an NxM matrix, where N equals the sample size and $M = K + 1$ where K is the order of the fit. For the problem to be meaningful we must have $M \ll N$. Also recall that $A = H(H^TH)^{-1}H^T$ is an NxN square symmetric matrix, and the matrix is related to the least square solution of

$$\underline{Z} = H\underline{a} + \underline{n} \tag{A-2}$$

where $\underline{Z}$ is the N dimensional measurement vector, $\underline{a}$ is an M dimensional coefficient vector, and $\underline{n}$ is the noise vector. The least square solution of Equation (A-2) yields:

$$\hat{\underline{a}} = (H^TH)^{-1}H^T\underline{Z} \tag{A-3}$$

and the best estimate of $\underline{Z}$ is given by

$$\hat{\underline{Z}} = H\hat{\underline{a}} = H(H^TH)^{-1}H^T\underline{Z} = A\underline{Z} \tag{A-4}$$

For a given matrix A, the notation $\mathrm{sum}[A] \triangleq \sum_{i}^{N} \sum_{j}^{N} A_{ij}$; i.e., the sum of all the elements in A. We want to show that given $A = H(H^T H)^{-1} H^T$, $\mathrm{sum}[A] = N$, where N is the dimension of A.

Using basic matrix manipulations, it is easy to establish the equivalence

$$\mathrm{sum}[A] = \underline{1}^T A \underline{1} \tag{A-5}$$

where $\underline{1}^T = [1\ 1\ 1\ \ldots\ 1]$, a row vector of N ones.

Denote the N dimensional real space by R^N, the M dimensional real space by R^M, then for all possible values of $\underline{Z}$, the measurement vector, and $\underline{n}$, the noise vector, the difference vector $\underline{d} = \underline{Z} - \underline{n}$ is contained in the space D^N which is a subspace of R^N; i.e., one can write

$$D^N = \{\underline{d} : \underline{d} = H\underline{a}\ \forall\ \underline{a} \epsilon R^M\}$$

The space D^N is a subspace of R^N because the Rank[H] < N. Denote the ith column of H by $\underline{h}_i$, then one can write

$$\begin{aligned} \underline{d} &= H\underline{a} \\ &= [\underline{h}_1\ \underline{h}_2\ \ldots\ h_M] \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_M \end{bmatrix} \\ &= \sum_{i=1}^{M} a_i \underline{h}_i \end{aligned} \tag{A-6}$$

Thus $\underline{d}$ is a linear combination of $\underline{h}_i$. In fact, elements of the set $\{\underline{h}_i\}$ i = 1, 2, ..., M are linearly independent and span the space D^N, therefore the set forms a basis. However, base vectors are not unique. In particular, there

exists an orthonormal basis; i.e., the base vectors are orthogonal to each other and have unity length. Given the basis $\{\underline{h}_i\}$, an orthonormal set can be constructed using the Grahm-Schmidt orthogonalization procedure.

$\| \ \|$ denotes the inner product norm and defines the following notations:

$$
\begin{aligned}
&\underline{E}_1 = \underline{h}_1 && ; \underline{e}_1 = \frac{\underline{h}_1}{\|\underline{h}_1\|} \\
&\underline{E}_2 = h_2 - (\underline{h}_2 \cdot \underline{e}_1)\underline{e}_1 && ; \underline{e}_2 = \frac{\underline{E}_2}{\|\underline{E}_2\|} \\
&\vdots && \vdots \\
&\underline{E}_M = \underline{h}_M - \sum_{i=1}^{M-1} (\underline{h}_M \cdot \underline{e}_i)\underline{e}_i && ; \underline{e}_M = \frac{\underline{E}_M}{\|\underline{E}_M\|}
\end{aligned}
\tag{A-7}
$$

then the set $\{\underline{e}_i\} = \left\{ \frac{\underline{E}_i}{\|\underline{E}_i\|} \right\}$ i = 1, 2, ..., M is the required orthonormal set. The relation between $\{\underline{e}_i\}$ and $\{\underline{h}_i\}$ can be established using Equation (A-6) since each $\underline{e}_i$ is a linear combination of the set $\{\underline{h}_i\}$. Thus, we have the relation

$$
[\underline{e}_1\ \underline{e}_2\ \cdots\ \underline{e}_M] = [\underline{h}_1\ \underline{h}_2\ \cdots\ \underline{h}_M] \begin{bmatrix} P_{11} & P_{21} & \cdots & P_{M1} \\ 0 & P_{22} & \cdots & P_{M2} \\ 0 & \cdot & \cdots & \cdot \\ \vdots & \vdots & & \vdots \\ 0 & P_{2M} & & P_{MM} \end{bmatrix}
$$

or in matrix form

$$
E = HP. \tag{A-8}
$$

Note $P_{11} = \frac{1}{\|\underline{h}_1\|}$, where $\|\underline{h}_1\| = N$ denotes the inner product norm of $\underline{h}_1$. Since $\{\underline{e}_i\}$ is an orthonormal set, we have the obvious relation

$$E^T E = I \tag{A-9}$$

Note that, since P is a basis transformation matrix, it is non-singular, and its inverse exists [4]. Post multiplying Equation (A-7) by P^{-1} yields

$$H = EP^{-1} \tag{A-10}$$

Substituting Equation (A-10) into Equation (A-3) yields

$$\begin{aligned} \hat{\underline{a}} &= (H^T H)^{-1} H^T \underline{Z} \\ &= [(P^{-1})^T E^T E P^{-1}]^{-1} (P^{-1})^T E^T \underline{Z} \\ &= P E^T \underline{Z} \end{aligned} \tag{A-11}$$

Now since $\underline{h}_1 = [1\ 1\ 1\ \ldots\ 1]^T = \underline{1}$, and $\underline{e}_1 = \frac{\underline{h}_1}{N}$, we have $\underline{1} = N\underline{e}_1$. Let $\underline{Z} = \underline{1}$ in Equation (A-11), we obtain

$$\hat{\underline{a}} = N \begin{bmatrix} P_{11} & P_{21} & \cdots & P_{M1} \\ 0 & P_{22} & \cdots & P_{M2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & P_{2M} & \cdots & P_{MM} \end{bmatrix} \begin{bmatrix} \underline{e}_1^T \underline{e}_1 \\ \underline{e}_2^T \underline{e}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{e}_M^T \underline{e}_1 \end{bmatrix}$$

$$= N \begin{bmatrix} \frac{1}{N} & P_{21} \cdots P_{M1} \\ 0 & P_{22} \cdots P_{M2} \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot \quad\quad \cdot \\ 0 & P_{2M} \cdots P_{MM} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{A-12}$$

Now using Equation (A-12) in Equation (A-4), we have the desired result

$$H\hat{\underline{a}} = \underline{1} \tag{A-13}$$

and

$$\begin{aligned} \text{sum}[A] &= \underline{1}^T A \underline{1} \\ &= \underline{1}^T H(H^T H)^{-1} H^T \underline{1} \\ &= \underline{1}^T H \hat{\underline{a}} \\ &= \underline{1}^T \underline{1} \\ &= N \qquad\qquad \text{Q.E.D.} \end{aligned}$$

APPENDIX B
PROOF OF $\mathrm{sum}[H(H^TH)^{-1}H^TRH(H^TH)^{-1}H^T] = \mathrm{sum}[R]$

Recall from Appendix A that the matrix H is given by Equation (A-1) and that R is the covariance matrix of the input sequence. Also recall that the notation sum[A] denotes the element sum of the matrix A.

Now

$$\mathrm{sum}[R] = \underline{1}^TR\underline{1}$$

where $\underline{1}^T = [1\ 1\ \ldots\ 1]$ a row vector of ones.

Then using results of Equations (A-1) and (A-13) from Appendix A, we obtain

$$\begin{aligned}
&\mathrm{sum}[H(H^TH)^{-1}H^TRH(H^TH)^{-1}H^T] \\
&= \underline{1}^TH(H^TH)^{-1}H^TRH(H^TH)^{-1}H^T\underline{1} \\
&= [H(H^TH)^{-1}H^T\underline{1}]^TR[H(H^TH)^{-1}H^T\underline{1}] \\
&= \underline{1}^TR\underline{1} \\
&= \mathrm{sum}[R]
\end{aligned}$$

since

$$H(H^TH)^{-1}H^T\underline{1} = \underline{1}$$

APPENDIX C

EQUIVALENT BANDWIDTH OF A FIRST ORDER DIGITAL FILTER

Given a first order digital filter

$$Y_i = \alpha Y_{i-1} + (1 - \alpha)X_i \tag{C-1}$$

with $i = 0, 1, 2, \ldots, 0 \leq \alpha \leq 1$ and $Y_i = 0$ for $i < 0$.

Taking the Z-transform of Equation (C-1) yields

$$Y(Z) = \alpha Z^{-1}Y(Z) + (1 - \alpha)X(Z) \tag{C-2}$$

and the transfer function is

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1 - \alpha}{1 - \alpha Z^{-1}} \tag{C-3}$$

The equivalent bandwidth of the transfer function is

$$BW_e = \frac{1}{|H(1)|^2} \frac{1}{2\pi j} \oint H(Z)H^*(Z)Z^{-1}dZ$$

$$= \frac{1}{2\pi j} \oint_{\substack{\text{unit} \\ \text{circle}}} H(Z)H^*(Z)Z^{-1}dZ \tag{C-4}$$

since $|H(1)|^2 = \left| \frac{1 - \alpha}{1 - \alpha Z^{-1}} \right|^2_{Z=1} = 1.$

Now $H(Z) = \dfrac{1 - \alpha}{1 - \alpha Z^{-1}}$ which has a pole at $Z_p = \alpha \leq 1$ and $H^*(Z) = \dfrac{1 - \alpha}{1 - \alpha Z}$ which has a pole at $Z_p' = \dfrac{1}{\alpha} \geq 1$, thus H(Z) has a pole inside the unit circle while H*(Z) has a pole outside the unit circle. Therefore using the residue theorem, we have

$$\frac{1}{2\pi J} \oint H(Z)H^*(Z)Z^{-1}dZ$$

$$= [(Z - Z_p)H(Z)H^*(Z)Z^{-1}]_{Z=Z_p}$$

$$= \left[\frac{(1 - \alpha)^2}{1 - \alpha Z}\right]_{Z=\alpha}$$

$$= \frac{(1 - \alpha)^2}{1 - \alpha^2}$$

$$= \frac{1 - \alpha}{1 + \alpha} \tag{C-5}$$

Therefore the equivalent bandwidth is

$$BW_e = \frac{1 - \alpha}{1 + \alpha} \cdot \frac{1}{\Delta t} \text{ (Hertz)} \tag{C-6}$$

where Δt is the sampling interval.

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-8